REPRESENTATION THEORY OF SPECIAL LINEAR ALGEBRA $s\ell(2,C)$ AND GENERATING FUNCTIONS INVOLVING APPELL'S FUNCTIONS

by

RITA BHATIA

Department of Mathematics
Indian Institute of Technology, Delhi

Submitted

in fulfilment of the requirements of the degree of
DOCTOR OF PHILOSOPHY
to the
Indian Institute of Technology, Delhi

June, 1979

ACKNOWLEDGEMENTS

I am extremely grateful to Dr. H.L. Manocha, Associate Professor, Department of Mathematics, Indian Institute of Technology, Delhi for his invaluable help, inspiring guidance and constant encouragement throughout the preparation of this thesis.

My thanks are due to Professor O.P. Bhutani, Head of the Department of Mathematics, Indian Institute of Technology, Delhi for his encouragement and interest in my work.

I am thankful to the Director, Indian Institute of Technology, Delhi for providing me all the facilities and research fellowship of the Institute.

I am very thankful to all my colleagues and specially to Dr. (Mrs.) Girija Jayaram, Miss Neelam Arora and Miss Poornima Dewanwala for their cooperation, ever willing help and encouragement.

My thanks are due to Mr. Arshad Majeed who gave his valuable time in reading the manuscript, making detailed corrections and making helpful suggestions.

I am extremely thankful to my husband Mr. K.B. Bhatia and my in-laws whose sacrifices and constant encouragement enabled me to complete my work.

Finally, I thank Mr. D.R. Joshi for his commendable typing.

(Rita Bhatia)

SYNOPSIS

The theory of Lie groups and Lie algebras plays a fundamental and key role in many branches of mathematics. Lie, Engel, Cartan, Killing and Weyl have been the great giants whose research works have led to its recent most elegant form which is very interesting and most useful in many applications of Mathematics.

For the past three decades mathematicians throughout the world have shown a keen interest in generating functions. Various techniques have been used by well known mathematicians to derive them. Representation theory of Lie groups and Lie algebras has been successfully applied notably by Weisner, Vilenkin and, more recently, by Miller, Kalnins and Manocha, to obtain important properties and generating functions of useful special functions which arise in many applications of mathematics. Lie groups have come to play an important role in modern physical theories and are used mostly because of their finite and infinite dimensional representations.

In the proposed thesis, representation theory of semi-simple Lie algebra $sl(2,C)$ has been used to show how and why important generating functions involving Appell's functions (Two variable hypergeometric functions) arise. Symmetry sub-algebras for second order partial differential equations in three variables are found out which are isomorphic to the special linear Lie algebra $sl(2,C)$. In fact, the separated solutions of certain linear equations are characterised as the common eigenfunctions of the sets of commuting elements in the universal enveloping algebra of the Lie symmetry sub-algebra corresponding to the equation. The local multiplier representation for the symmetry sub-algebra is determined which, together with the common eigen functions leads us to very powerful bilateral functions involving Appell's function $F_1(\alpha+n, \beta, \beta'; \gamma; x, y)$, $F_1(\alpha, \beta, \beta'+n; \gamma; x, y)$, $F_1(\alpha, \beta, \beta'; \gamma+n; x, y)$, $F_2(\alpha, \beta+n, \beta'; \gamma, \gamma'; x, y)$.

Details of the contents of various chapters are as follows:

In Chapter I, the main theory about Lie groups, Lie algebras and special functions required and used throughout the thesis is depicted in two parts. Part (a) contains Lie theory and part (b), the special function theory.

Chapter II: In this chapter we start with theorems providing us the symmetry algebra of a second order linear partial differential equation. It is shown that this symmetry algebra is isomorphic to special linear algebra $sl(2,C)$. The local multiplier representation for the symmetry algebra is determined. Further, a common eigen function of the two elements of the universal enveloping algebra is found, one of them being a Casimir operator which is in the centre of the universal enveloping algebra. The multiplier representation through this eigen-function enables us to obtain bilateral generating functions, believed to be new, involving the Appell's functions, $F_1[\alpha+n, \beta, \beta'; \gamma; x, y]$. Thereafter we derive a variety of special cases which are quite interesting.

Chapter III: This chapter deals with symmetry algebra of a second order linear partial differential equation whose solution is one of Saran's functions F_S. The method developed in Chapter II is followed for deriving bilateral generating functions for Appell's function $F_1[\alpha,\beta,\beta';\gamma+n;x,y]$. Also, a good many special cases are obtained.

Chapter IV: Symmetry algebra of a second order linear partial differential equation whose solution is F_M (one of Saran's hypergeometric function in three variables) is obtained. Symmetry algebra is isomorphic to special linear algebra $sl(2,C)$. The theory of $sl(2,C)$ is used to derive bilateral generating functions for $F_1[\alpha,\beta,\beta'+n;\gamma;x,y]$.

Chapter V: Methods developed in the preceding chapters are applied to obtain symmetry algebra, local multiplier representation, common eigen functions and finally generating functions for $F_2[\alpha,\beta+n,\beta';\gamma,\gamma';x,y]$ which with respect to a suitable basis arise as the matrix elements of an irreducible representation of $sl(2,C)$. Also, we obtain many important special cases.

Chapter VI: Generating functions for modified Bessel polynomials $Y_{m+n}^{(\gamma-2m-2n)}(x)$ have been obtained. The method in short consists of introducing partial linear differential operators using the differential recurrence relations satisfied by the given functions. Based on these operators local multiplier representation is determined. After finding the common eigen function, the generating functions for the given set of polynomials are obtained.

Chapter VII: In this chapter differential operator representation of orthogonal polynomials $P_n^{(\alpha-n,\beta-n)}(x)$ in term of a differential operator containing the generating function of the same is derived. The importance of the representation lies in the fact that we can obtain the linearly independent solutions of a class of homogeneous differential equations.

CONTENTS