HYPERGEOMETRIC SERIES IN THE p-ADIC SETTING, TRUNCATED HYPERGEOMETRIC SERIES AND SUPERCONGRUENCES

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by

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Dedicated to

My Family and Supervisor

Certificate

This is to certify that the thesis entitled **Hypergeometric Series in the p-Adic Setting, Truncated Hypergeometric Series and Supercongruences** submitted by **Ms. Neelam Saikia** to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

This thesis studies relations between hypergeometric series in the p-adic setting and algebraic curves; and certain supercongruences between truncated hypergeometric series and p-adic Gamma function. The theory of hypergeometric series over finite fields was developed in the mid 1980s by Greene. Since then, many mathematicians have studied connections between number of points on elliptic curves and hypergeometric series over a finite field. By definition, the arguments of hypergeometric series over finite fields are multiplicative characters of finite fields. Consequently, results involving hypergeometric series over a field are often restricted to primes in certain congruence classes to facilitate the existence of characters of specific orders. To overcome these restrictions, D. McCarthy defined a function ${}_nG_n[\cdots]$ in terms of quotients of the p-adic Gamma function which can best be described as an analogue of hypergeometric series in the p-adic setting. He showed how results involving hypergeometric series over a finite field can be extended to almost all primes using the function ${}_nG_n[\cdots]$. While numerous transformations exist for the finite field hypergeometric series, very few exist for ${}_nG_n[\cdots]$ in full generality. McCarthy posed an interesting question about finding transformations for ${}_nG_n[\cdots]$ which exist for all but finitely many p. In this thesis, by counting points on certain families of elliptic curves over finite fields, we derive certain transformations for ${}_2G_2[\cdots]$ which exist for all but finitely many p.

McCarthy showed that the trace of Frobenius of elliptic curves can be expressed in terms of the function ${}_2G_2[\cdots]$. In this thesis, we study the function ${}_nG_n[\cdots]$ for any $n \geq 2$. We present explicitly the number of distinct zeros of the polynomials x^n+ax+b and $x^n+ax^{n-1}+b$ over a finite field in terms of the function ${}_{n-1}G_{n-1}[\cdots]$ for any $n \geq 2$. We also express the number of points on the hyperelliptic curves $y^2 = x^n + ax + b$ and $y^2 = x^n + ax^{n-1} + b$ over a finite field in terms of the functions ${}_{n-1}G_{n-1}[\cdots]$ and ${}_{n-2}G_{n-2}[\cdots]$ for any $n \geq 2$. Finally, we establish several summation identities for ${}_nG_n[\cdots]$, where $n \geq 2$. We also deduce certain special values of the functions ${}_2G_2[\cdots]$ and ${}_3G_3[\cdots]$.

Various supercongruences between truncated hypergeometric series and p-adic Gamma function have been conjectured by many mathematicians including Beukers, van Hamme, Rodriguez-Villegas, Zudilin, Chan et al., and many more by Z.-W. Sun. Recently, L. Long and R. Ramakrishna prove several supercongruences using a technique which relies on the relations between the classical and p-adic Gamma functions. Using a similar technique we prove several more general supercongruences for truncated hypergeometric series and p-adic Gamma function. We also prove a supercongruence conjectured by A. Deines, J. Fuselier, L. Long, H. Swisher and F. Tu. A supercongruence conjectured by Rodriguez-Villegas and proved by E. Mortenson using the theory of Greene's finite field hypergeometric functions follows from one of our more general supercongruences.

Contents