UNITS IN FINITE LOOP ALGEBRAS OF RA AND $RA2$ LOOPS

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OF

RA AND $RA2$ LOOPS

by

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To

My RAM Ji

and

My Family

Certificate

This is to certify that the thesis entitled **"Units in Finite Loop Algebras of RA and $RA2$ loops"** submitted by **Ms. Swati Sidana** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.

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Abstract

A *loop* is a generalization of a group and an associative loop is called a group. Let R be an associative and commutative ring with identity. We construct the loop ring $R[L]$ from a loop L in a similar manner as the group ring $R[G]$ is constructed from a group G. If F is a field, then we call $F[L]$, a loop algebra and $F[G]$, a group algebra. The problem of determining the unit group of a group ring is very challenging and so is the problem to establish the structure of the unit loop of a loop ring or a loop algebra. In this thesis, we determine the structure of the unit loops of loop algebras of finite RA and $RA2$ loops over the finite fields.

An RA(*Ring Alternative*) loop is a loop whose loop ring $R[L]$ over some commutative, associative ring R with identity and of characteristic different from 2 is alternative but not associative. RA loops have been completely classified. The order of an RA loop is 2^n, for a natural number $n \geq 4$ and there are only seven non-isomorphic classes of RA loops. All RA loops are Moufang loops of the type $M(G, *, g_0)$, where G is a non-abelian group with involution $*$ on G and g_0 is an element in the center of G. We completely characterize the structure of the unit loop of the loop algebras of all RA loops of order 32, 64 and then in general, of all the seven classes of indecomposable RA loops over the finite fields of characteristic greater than 2. When $charF$ is 2 and L is an RA loop of order 2^n, then we prove that $\mathcal{U}(F[L]) \cong F^* \times (1 + \Delta_F(L))$ and $dim_F(\Delta_F(L)) = 2^n - 1$, where $\Delta_F(L)$ is the augmentation ideal of the loop algebra $F[L]$.

An $RA2$ loop is a loop whose loop ring in characteristic 2 is alternative but not associative. Every $RA2$ loop is an RA loop, but the converse is not true. All $RA2$ loops have not yet been classified. But CHEIN and GOODAIRE have studied a particular class of Moufang loop that forms $RA2$ loops. They proved that the Moufang loop of the type $M(G, 2)$ is an $RA2$ loop if G is one of the following groups:

- the symmetric group, S_3,
- the group of quaternions, Q_8,
- the dihedral group D_{2m},
- the generalized dihedral group $Dih(A)$ of any abelian group A, provided A has exponent > 2,
- any non-abelian group G of exponent 4 which has precisely two elements of order 2.

In this thesis, we deal with the loop algebras of $RA2$ loops $M(G, 2)$ obtained from the non-abelian groups mentioned above. We start with $M(S_3, 2)$ and determine the structure of the unit loop of its loop algebra over the finite fields of characteristic different from 3. The Wedderburn decomposition of $F[M(Q_8, 2)]$ when F is a finite field of characteristic different from 2 is already known. Next, we consider the loop algebra $F[M(D_{2m}, 2)]$ for an odd positive integer m and determine the structure of its unit loop over the field F of characteristic 2.

For any abelian group A, the *generalized dihedral group* of A is the semidirect product of A and C_2, with C_2 acting on A by inverting elements and is written as $Dih(A) = A \rtimes C_2$. For a prime p and a natural number m, we consider the generalized dihedral group of C_p^m and determine its matrix representations over the finite field F of characteristic 2 containing a primitive p^{th} root of unity. As a consequence, we explore the units in $F[M(Dih(C_p^m), 2)]$.

Contents

List of Tables

List of Symbols

Following is a list of symbols and other notations used in this thesis. In what follows, L is a loop, R is a ring and F is a field.

Symbol	**Meaning**
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\dot{\cup}$	disjoint union
$\forall$	for all
$comm(a, b)$	the commutator of elements a and b in L
$asso(a, b, c)$	the associator of elements a, b and c in L
$[a, b]$	the commutator of elements a and b in R
$[a, b, c]$	the associator of elements a, b and c in R
$\mathcal{N}_\lambda(L)$	the left nucleus of L
$\mathcal{N}_\rho(L)$	the right nucleus of L
$\mathcal{N}_\nu(L)$	the middle nucleus of L
$\mathcal{N}(L)$	the nucleus of L
$\mathcal{Z}(L)$	the center of L
$\mathcal{Z}(G)$	the center of a group G
L'	the commutator-associator subloop of L

ϕ	the Euler-phi function
$\Phi_n(x)$	the n^{th} cyclotomic polynomial
ξ_n	a primitive n^{th} root of unity
$gcd(a,b)$	the greatest common divisor of a and b
Q_8	the quaternion group of order 8
C_n	the cyclic group of order n
S_n	the symmetric group of degree n
D_{2n}	the dihedral group of order $2n$
$Dih(A)$	the generalized dihedral group of group A
$M(G,*,g_0)$	Moufang loop determined by a non-abelian group G with involution '$*$' and a central element g_0
$M(G,2)$	$M(G,-1,1)$, the special case of $M(G,*,g_0)$ where $*$ is the inverse map on G, and g_0 is the identity element
F^*	$F\backslash\{0\}$
$charF$	the characteristic of F
E/F	the field E is an extension of F
$[E:F]$	the degree of the field extension E/F
F_2	a quadratic field extension of F
F_k	a field extension of F of degree k
$GF(n)$	Galois field of order n
(F,α)	a two dimensional algebra obtained from F using Cayley-Dickson process
(F,α,β)	a generalized quaternion algebra over F
(F,α,β,γ)	a Cayley Dickson algebra over F
$M(n,R)$	the ring of $n \times n$ matrices over R
$GL(n,R)$	the General Linear group of degree n over R
$\mathcal{U}(R)$	the unit loop of R
$\mathfrak{Z}(R)$	Zorn's vector matrix algebra over R

$GLL(2,R)$	General Linear Loop, the loop of units of $\mathfrak{Z}(R)$
$R[L]$	the loop ring of L over R
$\mathcal{U}(R[L])$	the unit loop of loop ring $R[L]$
L/N	the quotient loop of L by N for a normal subloop N of L
ϵ_N	ring homomorphism $R[L] \to R[L/N]$
$\Delta_R(L,N)$	the kernel of ϵ_N
$\Delta_R(L)$	$\Delta_R(L,L)$, the augmentation ideal of the loop ring $R[L]$
$J(R)$	the Jacobson radical of R
$\widehat{N}$	$\sum\limits_{n\in N} n$, for a finite loop N contained in a loop ring
$\widetilde{N}$	$\frac{1}{\lvert N\rvert}\widehat{N}$, for a finite loop N contained in a loop ring $R[L]$ with $\lvert N\rvert$ invertible in R
$R \oplus S$	external direct sum of rings R and S
R^n	external direct sum of n copies of R
$L \times H$	external direct product of L and H
L^n	external direct product of n copies of L
$\prod\limits_{i\in I} L_i$	external direct product of family of loops $\{L_i \mid i \in I\}$
$L \rtimes H$	a semi direct product of L and H
$A \otimes_F B$	the tensor product of F-algebras A and B
$l.c.m.$	the least common multiple
$max\{i_1, i_2, \ldots, i_n\}$	maximum element among $i_1, i_2, \ldots, i_n$
$a \equiv b (\text{mod } n)$	$n \mid a - b$
$ord_n(q)$	the multiplicative order of q modulo n