A NEW APPROACH TO SIMPLE AND RELATED QUEUES

by
BOPPANA SHOBHA
Department of Mathematics

Submitted in fulfilment of the requirements
of the degree of
DOCTOR OF PHILOSOPHY

to the

INDIAN INSTITUTE OF TECHNOLOGY, DELHI

October, 1985

CERTIFICATE

It is certified that the thesis entitled "A New Approach to Simple and Related Queues" presented by Ms Boppana Shobha is worthy of consideration for the award of the degree of Doctor of Philosophy and is a record of the original bonafide research work carried out by her under my guidance and supervision and that the results contained in it have not been submitted in part or in full to any other university or institute for the award of any degree or diploma.

(Dr. O.P. Sharma) 1/X/85
Associate Professor
Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi-110016.

ACKNOWLEDGEMENTS

I would like to express my gratitude and appreciation to Prof. O.P. Sharma, who as my thesis supervisor provided encouragement, patient guidance and support throughout my research work. The large amount of time and painstaking efforts he devoted to my work are especially appreciated.

I express my deep sense of gratitude to Prof. O.P. Bhutani, Head of Mathematics Department and Prof. M.K. Jain, Deputy Director, I.I.T. Delhi for their constant encouragement throughout the preparation of this research work.

My thanks are also due to the authorities of I.I.T. Delhi for providing me with a scholarship and all the facilities during my research period.

Finally I thank Mr. D.R. Joshi for painstaking typing of the manuscript.

Shobha

(Boppana Shobha)

ABSTRACT

The queueing theory is a study of congestion and the problems falling under its purview can be broadly classified into two major catagories-one dealing with the transient behaviour and the second dealing with the steady state behaviour. The steady state solutions of a variety of queueing problems have been reported in the literature, but there is a large number of problems where the transient solutions have not been found and one falls an easy prey to study these problems with the help of their steady state solutions. But we know that many of these situations seldom reach the equilibrium state. On the contrary some of the operations start empty and then stop or are stopped at some specified time. Waiting space is another factor in the study of queueing models with practical applications. By taking infinite waiting space or infinite number of servers very elegant mathematical solutions have been obtained by the researchers but such solutions have limited practical applications. In this thesis, a new approach is developed to study the transient behaviour of certain queueing systems primarily with limited waiting spaces.

The thesis has been divided into four chapters. Brief details of these chapters are given below:

Chapter I: This chapter presents a concise introduction to queueing theory with more emphasis on the subject of the thesis, namely, simple Markovian queues. All definitions and concepts pertaining to the thesis are explained elaborately.

Chapter II: This chapter discusses the classical and supposedly one of the simplest systems of queueing theory whose solution in the literature is reported on the basis of one dimensional state model representing the number of units waiting in the system at a given time. Furthermore the known solution of this model for infinite waiting space is very complex in nature involving modified Bessel functions and infinite series of these functions. Consequently in many potential applications of this model quantitative estimates are obtained by using approximations of these functions or, worse still, the steady state results are used to describe the system contrary to the practical situations.

In this chapter, we study this classical but important problem with the help of a two dimensional state model. The two dimensions represent respectively the number of arrivals to and the departures from the system at a given time and their difference being the number waiting in the system at that time. Symbolically we take $p_{n,k}(r,t)$ to represent the probability that n customers arrive at and k leaves the system in $(0,t)$ with $r\ (=n-k)$ customers waiting at time t. A neat closed form solution is obtained as:

$$p_{r+k,k}(r,t) = e^{-\lambda t}\left[\frac{(\lambda t)^k}{k!} - \rho \frac{(\lambda t)^{k-1}}{(k-1)!} \right] \rho^r$$

$$+ e^{-(\lambda+\mu)t} \rho^r \frac{(\lambda t)^k}{k!} \sum_{m=0}^{r+k} (k-m) \frac{(\mu t)^{m-1}}{m!}$$

$$\rho = \lambda/\mu$$

so that

$$p(r,t) = \sum_{k=0}^{\infty} p_{r+k,k}(r,t) = \rho^r(1-\rho) + e^{-(\lambda+\mu)t} \rho^r \times$$

$$\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \sum_{m=0}^{r+k} (k-m) \frac{(\mu t)^{m-1}}{(m-1)!}$$

gives an alternative solution for the classical model which is computationally simpler than the one reported in the literature. The results for the finite waiting space and steady state case can be derived and various parameters of the system have been worked out.

Chapter III: In this chapter we study $M/M/c/N$ queueing system. The transient solution of $M/M/c/\infty$ is known and it has been pointed out in the literature that the study of transient behaviour with limited waiting space poses cumbersome difficulties even in the case of $M/M/1/N$ queue. We have succeeded in this chapter in obtaining the closed form and analytical solution of $M/M/c/N$ queue and this is expressed as:

For $\lambda \neq c\mu$,

$$
p(r,t) = \begin{cases}
\dfrac{(\rho c)^r (1-\rho)}{r!\left[(1-\rho)\displaystyle\sum_{j=0}^{c-1} \frac{(\rho c)^j}{j!} + \frac{(\rho c)^c}{c!}\left(1-\rho^{N+1-c}\right)\right]} + e^{-(\lambda + c\mu)t} \times \\[2ex]
\displaystyle\sum_{j=1}^{N} \frac{(-1)^r (\sqrt{\rho})^r e^{-\alpha_{N,j} t \sqrt{c\lambda\mu}}\, C_{N-r,j}}{\left(\sqrt{\rho} + \frac{1}{\sqrt{\rho}}\ \alpha_{N,j}\right) b_{N,j}} \\[2ex]
\qquad \text{for } r = 0,1,2,\ldots,c-2 \\[2ex]
\dfrac{c^c \rho^r (1-\rho)}{c!\left[(1-\rho)\displaystyle\sum_{j=0}^{c-1} \frac{(\rho c)^j}{j!} + \frac{(\rho c)^c}{c!}\left(1-\rho^{N+1-c}\right)\right]} + e^{-(\lambda + c\mu)t} \times \\[2ex]
\displaystyle\sum_{j=1}^{N} \frac{(-1)^r (\sqrt{\rho})^r e^{-\alpha_{N,j} t \sqrt{c\lambda\mu}} \left(a_{N-r,j} + \sqrt{\rho}\, a_{N-r-1,j}\right)}{\left(\sqrt{\rho} + \frac{1}{\sqrt{\rho}} + \alpha_{N,j}\right) b_{N,j}} \\[2ex]
\qquad \text{for } r = c-1, c, \ldots, N
\end{cases}
$$

and for $\lambda = c\mu$

$$
p(r,t) = \begin{cases}
\dfrac{c^r}{r!\left[\displaystyle\sum_{j=0}^{c-1} \frac{c^j}{j!} + \frac{c^c}{c!}(N+1-c)\right]} + e^{-2\lambda t} \times \\[2ex]
\dfrac{\displaystyle\sum_{j=1}^{N} (-1)^r e^{-\alpha_{N,j} t \lambda}\, C_{N-r,j}}{(2+\alpha_{N,j}) b_{N,j}}, \quad r = 0,1,\ldots,c-2 \\[2ex]
\dfrac{c^c}{c!\left[\displaystyle\sum_{j=0}^{c-1} \frac{c^j}{j!} + \frac{c^c}{c!}(N+1-c)\right]} + e^{-2\lambda t} \times \\[2ex]
\displaystyle\sum_{j=1}^{N} \frac{(-1)^r e^{-\alpha_{N,j} t \lambda} \left(a_{N-r,j} + a_{N-r-1,j}\right)}{(2 + \alpha_{N,j}) b_{N,j}} \\[2ex]
\qquad r = c-1, \ldots, N
\end{cases}
$$

$\lambda_{N,j}$; $b_{N,j}$ are suitably defined as functions of α_{Nj}, λ and μ.
It is eastablished that $P_N(x)$ has real and distinct roots. The steady state results come out directly by taking limit $t \to \infty$. Various system parameters have also been worked out. The results for $M/M/2/N$ model have been derived by taking $c = 2$.

<u>Chapter IV</u>: The busy period analysis for $M/M/1/\infty$ queue has been reported in the literature. In this chapter we carry out the analysis for $M/M/1/N$ queue and have obtained the density function for the busy period in the form

$$f(t) = e^{-(\lambda+\mu)t} \sum_{k=1}^{N} A_k \, e^{-\alpha_{Nk} t \sqrt{\lambda\mu}}$$

where α_{Nk} are the zeros of the nth degree polynomial

$$\sum_{k=0}^{[N/2]} (-1)^k \; {}^{N-k}C_k \, x^{N-2k} - \rho \sum_{k=0}^{[(N-1)/2]} (-1)^k \; {}^{N-k}C_k \, x^{N-1-2k},$$

$$\rho = \lambda/\mu$$

The various parameters of the busy period have been worked out. It is also found that the busy period distribution is the mixture of exponential distributions.

The analysis has been further extended to $M/M/2/N$ case and similar results for this case have also been obtained in closed form.

CONTENTS

CHAPTER		PAGE No.
	Certificate	i
	Acknowledgements	ii
	Contents	iii
	Abstract	iv - viii
I	Introduction	1 - 37
	1.1 General Description	2
	1.2 Poisson Queues	15
	1.3 Non-Poisson Queues	29
II	A New Approach to M/M/1 Queue	38 - 96
	2.1 Introduction	39
	2.2 Basic Equations and Calculations	42
	2.3a Infinite Waiting Space	50
	2.3b Marginal Distributions	64
	2.3c Conditional Probability Functions	73
	2.4 Finite Waiting Space	76
	Appendix	92
III	Transient Behaviour of M/M/c/N Queue	97 - 144
	3.1 Introduction	98
	3.2 Basic Equations and Main Results	100
	3.3 Some Important Parameters	134
IV	Busy Period Analysis for M/M/1/N and M/M/2/N Queues	145 - 180
	4.1 Introduction	146
	4.2a The Single Server Queue	148
	4.2b Important Parameters	160
	4.3 M/M/2/N Queues	163
	4.3a Two Server Busy Period	164
	4.3b Mean and Variance	170
	4.3c Single Server Busy Period	171
	References	181 - 205