RESULTS ON ABSTRACT MODULE CATEGORIES AND EILENBERG-MOORE CATEGORIES

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Results on Abstract Module Categories and Eilenberg-Moore Categories

by

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Department of Mathematics

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Dedicated to my parents

Mr. Vinod Kumar and Mrs. Shakuntla

Certificate

This is to certify that the thesis entitled **Results on Abstract Module Categories and Eilenberg-Moore Categories** submitted by **Ms. Divya** to **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
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Abstract

The goal of this thesis is to study the results on abstract module categories and Eilenberg-Moore Categories. This thesis is structured into two main parts. The first half focuses on the results on abstract module categories, beginning with a chapter that provides a brief survey of the literature on the subject.

In Chapter 2, we study local cohomology in the abstract module category, where we prove Grothendieck's vanishing theorem and Non-vanishing Theorem. In Chapter 3, we consider the categories of contramodule objects and comodule objects in a Grothendieck category over an entwining structure (A, C, ψ), where A is a k-algebra and C is a k-coalgebra. A measuring from an entwining structure (A', C', ψ') to (A, C, ψ) is considered, which induces pairs of adjoint functors between the categories of entwined contramodule objects as well as the categories of entwined comodule objects. We provide conditions under which these adjoint pairs are inverse equivalences. In this chapter, we also study separability, Frobenius, and Maschke-type results for functors between categories of entwined comodules and entwined contramodules.

The second half of the thesis focuses on the results on Eilenberg-Moore categories. We begin this part with a review of the literature on Eilenberg-Moore categories in Chapter 4. In Chapter 5, we consider representations of quiver taking values in the category of monads and comonads. We develop a categorical framework for modules and comodules over these representations. Our goal is to give conditions under which these categories become Grothendieck. We also study the categories of cartesian modules and comodules over these representations, which resemble quasi-coherent sheaves. We conclude this chapter by discussing the rational pairing between monad and comonad representations.

In Chapter 6, we consider a differential monad (U, θ, η, d) on a Grothendieck category, where $d : U \longrightarrow U$ is a derivation on U, and we define a d-derivation D on a module over the monad

U. We prove in this chapter that every hereditary differential torsion theory on the Eilenberg-Moore category of modules over a monad (U, θ, η, d) is differential. Further, we show that every d-derivation on a module in the Eilenberg-Moore category can be extended to the module of quotients.

सारांश

यह शोध प्रबंध का उद्देश्य अमूर्त मॉड्यूल श्रेणियों और आयलेनबर्ग-मूर श्रेणियों पर परिणामों का अध्ययन करना है। यह शोध प्रबंध मुख्य रूप से दो भागों में विभाजित है। पहले भाग में अमूर्त मॉड्यूल श्रेणियों पर आधारित परिणामों पर ध्यान केंद्रित किया गया है, जिसकी शुरुआत अध्याय १ से होती है, जिसमें इस विषय पर साहित्य का संक्षिप्त सर्वेक्षण दिया गया है।

अध्याय २ में, हम अमूर्त मॉड्यूल श्रेणी में स्थानीय कोहोमोलॉजी का अध्ययन करते हैं, जहाँ हम ग्रोथेंडीक का वैनिशिंग थ्योरम और नॉन-वैनिशिंग थ्योरम प्रमाणित करते हैं। अध्याय ३ में, हम ग्रोथेंडीक श्रेणी में एक एंट्वाइनिंग संरचना (A, C, ψ), जहाँ A एक k-बीजगणित है और C एक k-कोलबीजगणित है, पर आधारित कंट्रामॉड्यूल और कोमॉड्यूल वस्तुओं की श्रेणियों पर विचार करते हैं। एक मापन (A', C', ψ') को (A, C, ψ) तक परिभाषित किया जाता है, जो एंट्वाइन कंट्रामॉड्यूल वस्तुओं और एंट्वाइन कोमॉड्यूल वस्तुओं की श्रेणियों के बीच युग्म समांतर फंक्टर प्रेरित करता है। हम उन परिस्थितियों को प्रदान करते हैं, जिनके तहत ये युग्म समांतर युग्म प्रतिलोम समकक्षाएँ बन जाते हैं। इस अध्याय में, हम एंट्वाइन कोमॉड्यूल्स और कंट्रामॉड्यूल्स की श्रेणियों के बीच फंक्टरों के लिए पृथक्करण, फ्रॉबेनीयस , और मास्के-प्रकार के परिणामों का भी अध्ययन करते हैं।

शोध प्रबंध का दूसरा भाग आयलेनबर्ग-मूर श्रेणियों पर आधारित परिणामों पर केंद्रित है। हम इस भाग की शुरुआत अध्याय ४ में आयलेनबर्ग-मूर श्रेणियों पर साहित्य समीक्षा से करते हैं। अध्याय ५ में, हम क्विवर के उन रूपों का अध्ययन करते हैं, जो मॉनाड और कोमॉनाड्स की श्रेणी में मान ग्रहण करते हैं। हम मॉड्यूल्स और कोमॉड्यूल्स के लिए एक श्रेणीगत ढांचा विकसित करते हैं और उन स्थितियों का वर्णन करते हैं, जिनके तहत ये श्रेणियाँ ग्रोथेंडीक बन जाती हैं। हम इन अभ्यावेदन पर आधारित कार्टेशियन मॉड्यूल्स और कोमॉड्यूल्स की श्रेणियों का भी अध्ययन करते हैं, जो लगभग अर्ध-सुसंगत के समान होती हैं। हम इस अध्याय का समापन मॉनाड और कोमॉनाड अभ्यावेदन के बीच युक्तियुक्त युग्मन पर चर्चा करके करते हैं।

अध्याय ६ में, हम ग्रोथेंडीक श्रेणी पर एक अवकलन मॉनाड (U, θ, η, d) पर विचार करते हैं, जहाँ पर $d : U \to U$ एक व्युत्पन्न है, और हम मॉनाड U पर एक मॉड्यूल पर d-व्युत्पन्न परिभाषित करते हैं। इस अध्याय में, हम प्रमाणित करते हैं कि मॉनाड (U, θ, η, d) पर मॉड्यूल की आयलेनबर्ग-मूर श्रेणी में हर वंशानुगत अवकलन टॉर्शन सिद्धांत अवकलन होता है। इसके अलावा, हम दिखाते हैं कि आयलेनबर्ग-मूर श्रेणी में मॉड्यूल पर हर d-व्युत्पन्न को भागांक तक विस्तारित किया जा सकता है।

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List of Symbols

Symbol	Meaning
$\forall$	for all
$\exists$	there exists
$\emptyset$	empty set
$\oplus$	coproduct
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q} = (\mathbb{V}, \mathbb{E})$	Quiver; a directed graph with a set of vertices $\mathbb{V}$ and a set of edges $\mathbb{E}$
k	a field of characteristic zero
R	an associative ring with unity
η	unit map
μ, θ	multiplication map
Δ, δ	comultiplication map
ϵ	counit map
(A, μ, η)	a k algebra
(C, Δ, ϵ)	a k-coalgebra
(U, θ, η)	monad
(V, δ, ϵ)	comonad
$\otimes_k$	tensor product of k-algebra
$\mathrm{Ker}(f)$	Kernel of a morphism f
$\mathrm{Im}(f)$	Image of a morphism f
$\mathrm{Coker}(f)$	Cokernel of a morphism f
${}_A\mathrm{Mod}$	the category of left A-modules

Mod_A	the category of right A-modules
Vect_k	the category of vector spaces over k
EM_U	Eilenberg-Moore category of modules over the monad U
EM^V	Eilenberg-Moore category of comodules over the comonad V