

**TWISTED DERIVATIONS IN GROUP RINGS AND
ALGEBRAIC NUMBER FIELDS WITH CODING THEORY
APPLICATIONS**

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JULY 2024**

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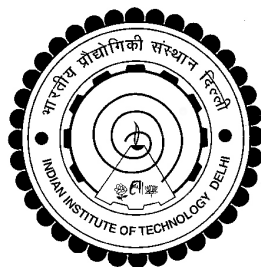
by

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Submitted

*in fulfillment of the requirements of the degree of Doctor of Philosophy
to the*



INDIAN INSTITUTE OF TECHNOLOGY DELHI
JULY 2024

Dedicated to my Parents and my Teachers

Certificate

This is to certify that the thesis entitled **Twisted Derivations in Group Rings and Algebraic Number Fields with Coding Theory Applications** submitted by **Ms. Praveen Manju** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
July 2024

Prof. Rajendra Kumar Sharma
Department of Mathematics
Indian Institute of Technology Delhi

Acknowledgments

This thesis goes beyond a dissertation. This serves as evidence and a representation of my unwavering devotion to, deep love, and enthusiasm for my field of study, algebra. Since my graduation, I have not only studied algebra but have lived it each day to the fullest and cherished every minute of it. And I am grateful to God for blessing me with this beautiful gift of getting a doctorate in it. Therefore, first and foremost, I thank the almighty God for always respecting my faith in him. His presence in my heart strengthened me and helped me keep moving towards my dreams.

I express my gratitude to Prof. Rajendra Kumar Sharma, my Ph.D. supervisor, for his unwavering guidance and support in ensuring the success of this journey. I genuinely admire his experienced guidance and calm nature. He never prevented me from following and pursuing my desires and dreams, whether attending a new course, a conference, a workshop, a seminar, or doing the work at my own pace. This has helped me reach my highest potential in every possible way: becoming a competent researcher and teacher. There were days and moments when I doubted my capability, but he believed in me and put his complete confidence in me and my abilities by saying that one day, I would be coming out with flying colors. He always believed in me, and I am highly thankful to God for making him my supervisor. I owe him all the qualities that I have gained as a researcher. I also want to thank Prof. Ritumoni Sarma, my SRC (Student Research Committee) chairperson, for his help and advice over the years. Since I have a strong interest in algebra, I have always been excited to TA algebra-focused courses. However, with so many Ph.D. students in the department, it took a lot of work to fulfill each student's requirements. However, I am grateful to him for allowing me to teach linear algebra in MTL502. I will never forget the several occasions he has dedicated his invaluable time to me, inspiring and advising me. Whenever I needed his professional advice, he was always there. I am grateful to him for everything he has done to help me complete my doctorate successfully.

I also thank my SRC expert from the department, Prof. Amit Priyadarshi, for his kind and expert guidance. I asked him a few general queries a few times, and he answered them patiently and gracefully. I also recall the last semester of my Ph.D. when I wanted to do a TA for a different course, which I had yet to do until then, and I approached him for the TA for Measure and Integration. I am highly thankful to him for agreeing to give me this opportunity by putting his trust and belief in my sincerity. But later, I was granted TA of MTL105 (Algebra), my first preference from the list of courses. I also thank my SRC expert from outside the department, Prof. R. K. Varshney. Further, I thank other faculty members and staff of the department who directly and indirectly helped me during my journey. With special gratitude, I acknowledge Prof. Sivananthan Sampath, whom I have consulted on some occasions and who has generously shared his professional wisdom. Sir has consistently recognized my passion and enthusiasm for my subject. I am also grateful to Prof. Tanusree Chakraborty for organizing various yoga and meditation sessions, which helped me overcome extreme stress. I am thankful to the Central Library of IIT Delhi for offering a comfortable and conducive location and environment in which to sit. I have often sat there quietly to conduct research and find peace of mind. I also want to thank the Central Library team for hosting several workshops on publishing in research and ethics, which have been very helpful to me in learning about the field of research. I finally thank the entire IIT Delhi for providing me with a good study environment, academic opportunities, and research platform.

I am grateful to all my teachers from my school days who have always motivated me to excel and succeed. I thank my Math teacher, Anju Ma'am; Hindi teacher, Amita Pandey Ma'am; Social Science teacher, Ragini Ma'am; Science teacher, Kanchan Ma'am, for all their teachings. I remember that in my classes, 11th and 12th, my Math teacher, Gauri Shankar Sir, used to remark, "One day, we will be saluting you." His tremendous faith has always been my strength. I also remember that Pandey Sir, my Physics teacher, used to tell me to "never lose courage and keep working hard." This has always helped me to get back again and fight for my dreams whenever I fell. I remember my Biology teacher, Budh Ghosh Sir, who always believed in me; my Chemistry teacher, Jadon Sir, who had tremendous faith in me; and my English teacher, Om Prakash, Sir, who used to tell my father that he was lucky to have such a daughter. I am grateful to Monika Ma'am, my graduation teacher, who inspired me by showing me the beauty of algebra and helping me fall in love with it. I also appreciate Meetu Ma'am, Jyoti Ma'am, Apeksha Ma'am, Rekha Ma'am, Daulti Ma'am, Shashi Ma'am, Sunila Ma'am, Manjari Ma'am, Ritu Ma'am, Naveen Sir, Mohd Sarik Sir, and Vikas Sir for all the mathematical abilities I received during my graduation. I am grateful to my post-graduation instructors,

particularly Anuj Bishnoi Sir and Atul Gaur Sir, to whom I owe most of my algebraic abilities. I thank Ajay Kumar Sir and Ratikanta Panda Sir for their extraordinary teachings and guidance during my M.Sc. days. I also thank my other M.Sc. teachers, Randheer Sir, Ruchi Ma'am, Surender Sir, Lalit Sir, Lalitha Ma'am, and Ambethkar Sir, for imparting me knowledge of several mathematical subfields. I thank wholeheartedly every teacher whose name I may have forgotten and have not mentioned for making me who I am today.

I express my gratitude to my mother, Smt. Kamlesh Devi, who has consistently served as an inspiration and role model for me. She has been my closest companion during this adventure. The fact that I could live at home with her as a day scholar is a blessing. Going home and telling her everything that transpired during the day usually helped to reduce and relieve my stress after a long day at work. Despite teaching me many home skills, she kept me from applying them. She constantly wanted me to concentrate on my studies, taking care of the rest herself. She cared for my health and made me breakfast, lunch, and dinner every day of my Ph.D. During my stressful days, her unwavering moral support and guidance always gave me the strength to get back up and keep fighting without giving up. Her trust and confidence in me have been my strength. She is the primary factor in my doctorate's successful completion. I am so grateful to my mother that I want her to be my mother in every birth. I express my gratitude to my father, Sh. Rajesh Kumar, for taking on the responsibility of driving me to and from IIT each morning and evening. His assistance in arranging for me to attend national and international conferences and seminars has been invaluable in helping me realize my goal of finishing this thesis. I will never forget his concern when I wanted to attend a workshop at HRI, Allahabad, so he rode with me on the train to drop me off. He even came by train to fetch me up from there. I will never forget how he always made it easy and convenient for me to travel to conferences and workshops. My most significant source of inspiration and strength has come from my parents' unwavering work ethic and belief in me. They never impeded my pursuit of my aspirations and ambitions. I am particularly grateful to my younger brother Himanshu, whose cheerful and fun demeanor always brings me joy. Especially on stressful days, I would go on walks and play badminton with him. Whenever I had to choose between two things, I asked him for wise counsel.

I thank my maternal grandmother, Late Smt. Ompati Devi, who always motivated me to shine. Her belief in me and her inspirational advice in her sweet voice has always been there in my heart. But I had to go through the unimaginable pain of losing her in May last year when she lost her life as a result of lung failure. She was waiting for the day when I would get a

job, and I would buy a car with my own money. She would sit in my car, and I would show beautiful places and temples to her across the country. But unfortunately, I lost her before I could make this dream come true. But her dream always motivated me to succeed during my Ph.D. I will never accept defeat until I make her dream a reality one day. I thank my maternal grandfather, Sh. Shishram Maan, whose extraordinary life of respect and struggles has always inspired me to achieve everything through hard work, honesty, and sincerity. He has always motivated me to work hard constantly towards growth and success. He has always taken great pride in my achievements. I also thank my maternal uncles, aunts, and cousins for their belief and support.

Lastly, I would like to thank my seniors, juniors, batchmates, friends, and well-wishers who directly or indirectly assisted me in my journey. I want to thank Nitin Sir for his assistance with MATLAB while working on a research paper that included a conjecture I wished to confirm for a specific set of primes. I would especially like to thank Sushmita, who consistently offered me support and wise counsel whenever I reached out to her. Whenever I needed her, she was there for me. I would also like to express my gratitude to Abhilash, Anshul, Ritika, Sonamani, and Divya for their guidance and for accompanying me for a cup of tea or coffee during my difficult moments. I also thank Gurdev and Sachin for their help and advice during my Ph.D. I also acknowledge the power of music throughout my PhD journey. Music motivated me and worked as a source of meditation for me.

This thesis results from my immense hard work, dedication, perseverance, and persistence throughout these five years of my journey. I have lost motivation several times and fought many silent battles during these years, but each time, I have always been able to get back up and keep fighting for my goals. My desire to perform well in all my tasks and responsibilities has always caused me to suffer. I am happy to report that I have done rigorous coursework, given my best during my Teaching Assistantship, and attended and participated in various academic events, including conferences, workshops, seminars relevant to my research area, and numerous other academic gatherings that piqued my interest. Not to mention, I have worked hard in my primary research field. However, this is only the start of my wonderful experience as an algebraic researcher.

Abstract

In this thesis, we study twisted derivations in group rings and algebraic number fields. The theory of derivations has been developed in rings and various algebras and helps to study and understand their structure. Derivations play an essential role in mathematics and physics. They are extensively investigated in physics and engineering. The motivation to study twisted derivations in group rings and algebraic number fields comes from the fact that twisted derivations have vast applications in mathematics, physics, other sciences, and social sciences and have received little attention in these algebraic structures. Therefore, this thesis aims to contribute to studying twisted derivations in these two critical algebraic structures, namely, group rings and algebraic number fields. Also, applications of twisted derivations in these algebraic structures have been explored in Coding Theory. The thesis assumes the reader's knowledge of group, ring, field, module, basic structure theory of these algebraic structures, and basic coding theory. The thesis has been divided into ten chapters.

Let R be a commutative ring with unity, and \mathcal{A} and \mathcal{B} be R -algebras with $\mathcal{A} \subseteq \mathcal{B}$. Let $\sigma, \tau : \mathcal{A} \rightarrow \mathcal{B}$ be two R -algebra homomorphisms. A (σ, τ) -derivation or twisted derivation $D : \mathcal{A} \rightarrow \mathcal{B}$ is an R -linear map that satisfies the twisted generalized identity: $D(\alpha\beta) = D(\alpha)\tau(\beta) + \sigma(\alpha)D(\beta)$ for all $\alpha, \beta \in \mathcal{A}$. It is called inner if there exists some $\gamma \in \mathcal{B}$ such that $D(\alpha) = \gamma\tau(\alpha) - \sigma(\alpha)\gamma$ for all $\alpha \in \mathcal{A}$. If D is not inner, then it is called outer. When $\sigma = \tau$, then a (σ, τ) -derivation, inner (σ, τ) -derivation, and outer (σ, τ) -derivation are simply called σ -derivation, inner σ -derivation, and outer σ -derivation, respectively. When $\sigma = \tau = \text{id}_{\mathcal{A}}$, then a (σ, τ) -derivation satisfies the usual Leibniz rule: $D(\alpha\beta) = D(\alpha)\beta + \alpha D(\beta)$ for all $\alpha, \beta \in \mathcal{A}$, and then a (σ, τ) -derivation, inner (σ, τ) -derivation, and outer (σ, τ) -derivation are called derivation, inner derivation, and outer derivation, respectively. Analogously, we define these concepts for the twisted derivations of an arbitrary ring R . But in this case, we do not consider the R -linearity condition. Here, D must only be additive and satisfy the twisted Leibniz rule.

In Chapter 1, we introduce the work carried out in the thesis. We discuss the ABT structure of the thesis. “A” stands for “and,” which signifies what is already known in literature. This can be found in sections 1.1, 1.2, 1.3 and 1.4 of Chapter 1. In Section 1.1, we discuss the motivation behind picking up this topic. Section 1.2 discusses the basic definitions required to understand the developed theory. Section 1.3 discusses the literature on twisted derivations in group rings. In Section 1.4, we discuss the literature available on twisted derivations in algebraic number fields. “B” stands for “but,” which signifies the gaps in the existing literature that need to be filled and unknowns that need to be discovered. Finally, “T” stands for “therefore,” which signifies what we aim to do further. “B” and “T” have been discussed in Section 1.5 of Chapter 1, in which we discuss the research problems worked upon in this thesis.

In Chapter 2, we discuss the preliminaries required in the remaining chapters of the thesis. In Section 2.1, we state some necessary lemmas on twisted derivations. In Section 2.2, we give some known results from group theory such as results on conjugacy classes of some crucial groups (for example, dihedral group D_{2n} , dicyclic group T_{4n} , semi-dihedral group SD_{8n} and V_{8n}), universal property of free groups, doubly-twisted conjugacy classes and some results on them, group homomorphisms from D_{2m} to D_{2n} . In Section 2.3, we state some well-known definitions and results from the theory of group rings, such as the augmentation ideal, Maschke’s theorem, Wedderburn-Artin Theorem, class sums, the concept of group ring matrices, and various exciting and valuable results concerning these group matrices. In Section 2.4, we discuss some results from the theory of linear algebra over commutative rings. In Section 2.5, we state basic definitions and results from algebraic number theory. In Section 2.6, we discuss some basic definitions from coding theory and some known results on zero-divisor and unit-derived codes from group rings.

In Chapter 3, we study the derivations of group algebras of some influential groups, namely, Dihedral (D_{2n}), Dicyclic (T_{4n}) and Semi-dihedral (SD_{8n}). First, we explicitly classify all inner derivations of a group algebra $\mathbb{F}G$ of a finite group G over an arbitrary field \mathbb{F} . Then we classify all \mathbb{F} -derivations of the group algebras $\mathbb{F}D_{2n}$, $\mathbb{F}T_{4n}$ and $\mathbb{F}(SD_{8n})$ when \mathbb{F} is a field of characteristic 0 or an odd rational prime p by giving the dimension and an explicit basis of these derivation algebras. We explicitly describe all inner derivations of these group algebras over an arbitrary field. Finally, we classify all the above group algebras derivations when \mathbb{F} is an algebraic extension of a prime field. In particular, when \mathbb{F} is an algebraic extension of a prime field, we give a sufficient condition under which these group algebras have a non-trivial outer derivation, thus answering the derivation problem in these group algebras.

In Chapter 4, we study inner and outer derivations of the group algebra $\mathbb{F}V_{8n}$ of V_{8n} , a group of order $8n$ (n a positive integer) with presentation $\langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle$ over a field \mathbb{F} of characteristic 0 or an odd rational prime p . We explicitly classify all the inner and outer derivations of the group algebra $\mathbb{F}V_{8n}$. First, we explicitly classify all the \mathbb{F} -derivations of $\mathbb{F}V_{8n}$ by giving the dimension and a basis of the derivation algebra consisting of all \mathbb{F} -derivations of $\mathbb{F}V_{8n}$. Consequently, we classify all inner and outer derivations of $\mathbb{F}V_{8n}$ when \mathbb{F} is an algebraic extension of a prime field. Thus, we establish that all the derivations of $\mathbb{F}V_{8n}$ are inner when the characteristic of \mathbb{F} is 0 or p with p relatively prime to n , and that non-zero outer derivations exist only in the case when the characteristic of \mathbb{F} is p with p dividing n .

In Chapter 5, we study (σ, τ) -derivations of group rings in terms of generators and relators of the group. Leo Creedon and Kieran Hughes in [25] studied derivations of group ring RG (R a commutative unital ring) in terms of generators and relators of group G . In this chapter, we do that for (σ, τ) -derivations. We develop a necessary and sufficient condition such that a map $f : X \rightarrow RG$ can be extended uniquely to a (σ, τ) -derivation D of RG , where R is a commutative ring with unity, G is a group having a presentation $\langle X \mid Y \rangle$ (X the set of generators and Y the set of relators) and (σ, τ) is a pair of endomorphisms of RG which are R -linear extensions of the group homomorphisms of G . Further, we classify all inner (σ, τ) -derivations of the group algebra $\mathbb{F}G$ of a finite group G over an arbitrary field \mathbb{F} in terms of the dimension and a basis of the corresponding \mathbb{F} -vector space consisting of all inner (σ, τ) -derivations of $\mathbb{F}G$. We also prove that if R is a unital ring and G is a group whose order is invertible in R , then every (σ, τ) -derivation of RG is inner. We finally apply the results obtained above to classify all inner and outer σ -derivations of dihedral group algebras $\mathbb{F}D_{2n}$ ($D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, $n \geq 3$) over an arbitrary field \mathbb{F} of any characteristic, thus answering the twisted derivation problem in dihedral group algebras.

In Chapter 6, we study inner and outer twisted or (σ, τ) -derivations of the group ring RG of a finite cyclic group G over a commutative ring R with unity. Let R be a commutative ring with unity, G be a finite cyclic group, and (σ, τ) be a pair of R -algebra endomorphisms of the group algebra RG , which are R -linear extensions of the group endomorphisms of G . We give two characterizations concerning (σ, τ) -derivations of the group ring RG . First, we provide a necessary and sufficient condition for an R -linear map $D : RG \rightarrow RG$ with $D(1) = 0$ to be a (σ, τ) -derivation. Second, we develop a necessary and sufficient condition for a (σ, τ) -derivation of RG to be inner. We also illustrate our theorems with the help of examples.

As a consequence of these two characterizations, we answer the well-known twisted derivation problem for RG : Under what conditions are all (σ, τ) -derivations of RG inner? Or is the space of outer twisted derivations trivial? More precisely, we give a sufficient condition under which all (σ, τ) -derivations of RG are inner and a sufficient condition under which RG has non-trivial outer (σ, τ) -derivations. Our result helps in generating several examples of non-trivial outer (σ, τ) -derivations.

In Chapter 7, we study (σ, τ) -derivations of number rings by considering them as commutative unital \mathbb{Z} -algebras. We begin by characterizing all (σ, τ) -derivations and inner (σ, τ) -derivations of the ring of algebraic integers of a quadratic number field. Then we characterize all (σ, τ) -derivations of the ring of algebraic integers $\mathbb{Z}[\zeta]$ of a p^{th} -cyclotomic number field $\mathbb{Q}(\zeta)$ (p odd rational prime and ζ a primitive p^{th} -root of unity). We also conjecture (using SAGE and MATLAB) an “if and only if” condition for a (σ, τ) -derivation D on $\mathbb{Z}[\zeta]$ to be inner. We further characterize all (σ, τ) -derivations and inner (σ, τ) -derivations of the bi-quadratic number ring $\mathbb{Z}[\sqrt{m}, \sqrt{n}]$ (m, n distinct square-free rational integers). In each of the above cases, we also determine the rank and an explicit basis of the derivation algebra consisting of all (σ, τ) -derivations of the number ring.

In Chapter 8, we study twisted derivations in algebraic number fields. Let A be a commutative ring with unity, $B = A[\theta]$ (for some $\theta \in B$ integral over A) be an integral domain with quotient field \mathbb{K} , and \mathbb{E} be the minimal splitting field of θ over \mathbb{K} . Suppose $\sigma, \tau : B \rightarrow \mathbb{E}$ are ring homomorphisms that fix A element-wise. We prove that an A -linear map $D : B \rightarrow \mathbb{E}$ with $D(1) = 0$ satisfying certain conditions is a (σ, τ) -derivation. Consequently, we obtain results on twisted derivations in algebraic number fields and their ring of algebraic integers. For the ring of algebraic integers, $O_{\mathbb{K}} = \mathbb{Z}[\zeta]$ of the cyclotomic number field $\mathbb{K} = \mathbb{Q}(\zeta)$ (ζ an n^{th} primitive root of unity), and a pair (σ, τ) of two different \mathbb{Z} -algebra endomorphisms of $O_{\mathbb{K}}$, we conjecture a necessary and sufficient condition for a (σ, τ) -derivation $D : O_{\mathbb{K}} \rightarrow O_{\mathbb{K}}$ to be inner. This is done for two different forms of n : (i) $n = 2^r p$ ($r \in \mathbb{N}$ and p an odd rational prime), and (ii) $n = p^k$ ($k \in \mathbb{N}$ and p any rational prime). We also conjecture the existence and non-existence of non-zero outer derivations for the above two forms of n , thus answering the twisted derivation problem.

In Chapter 9, we study the applications of the work done in the previous six chapters from 3 to 8 in algebraic coding theory. We construct good parameter codes, some of which have additional properties, such as optimal, self-dual, self-orthogonal, and LCD. We give the notion

of an “image of derivation-derived code”. Also, we show that derivations occur as zero-divisor and unit-derived codes.

Finally, in Chapter 10, we conclude all the thesis findings by highlighting the main contributions of the thesis. We also give several future research directions arising out of the work carried out in this thesis.

सार

इस थीसिस में, हम ग्रुप रिंग्स और अलजेब्राइक नंबर फ़िल्ड्स में ट्विस्टेड डेरिवेशन का अध्ययन करते हैं। डेरिवेशनस का सिद्धांत रिंग्स और विभिन्न अलजेबराज़ में विकसित किया गया है और उनकी संरचना का अध्ययन करने और समझने में मदद करता है। डेरिवेशनस गणित और भौतिकी में एक महत्वपूर्ण भूमिका निभाती हैं। भौतिकी और इंजीनियरिंग में उनकी व्यापक रूप से जांच की जाती है। ग्रुप रिंग्स और अलजेब्राइक नंबर फ़िल्ड्स में ट्विस्टेड डेरिवेशनस का अध्ययन करने की प्रेरणा इस तथ्य से मिलती है कि ट्विस्टेड डेरिवेशनस के गणित, भौतिकी, अन्य विज्ञानों और सामाजिक विज्ञानों में विशाल अनुप्रयोग हैं और इन बीजगणितीय संरचनाओं में बहुत कम ध्यान दिया गया है। इसलिए, इस शोध प्रबंध का उद्देश्य इन दो महत्वपूर्ण बीजगणितीय संरचनाओं, अर्थात् ग्रुप रिंग्स और अलजेब्राइक नंबर फ़िल्ड्स में ट्विस्टेड डेरिवेशनस का अध्ययन करने में योगदान करना है। इसके अलावा, इन बीजगणितीय संरचनाओं में ट्विस्टेड डेरिवेशनस के अनुप्रयोगों को कोडिंग सिद्धांत में खोजा गया है। थीसिस ग्रुप, रिंग, फ़िल्ड, मॉड्यूल, इन बीजगणितीय संरचनाओं के बुनियादी संरचना सिद्धांत और बुनियादी कोडिंग सिद्धांत के बारे में पाठक के ज्ञान को मानता है। इस शोध प्रबंध को दस अध्यायों में विभाजित किया गया है।

मान लीजिए कि R यूनिटी के साथ एक कम्यूटेटिव रिंग है, और \mathcal{A} और B , $\mathcal{A} \subseteq B$ के साथ R -अल्जेब्रा हैं। मान लीजिए $\sigma, \tau: \mathcal{A} \rightarrow B$ दो R -अल्जेब्रा होमोमोर्फिजम्स हैं। एक (σ, τ) -डेरिवेशन या ट्विस्टेड डेरिवेशनस $D: \mathcal{A} \rightarrow B$ एक R -लीनियर मैप है जो ट्विस्टेड सामान्यीकृत पहचान को संतुष्ट करता है: $D(\alpha\beta) = D(\alpha)\tau(\beta) + \sigma(\alpha)D(\beta)$, सभी $\alpha, \beta \in \mathcal{A}$ के लिए। इसे इनर कहा जाता है यदि कुछ $\gamma \in B$ मौजूद है ताकि $D(\alpha) = \gamma\tau(\alpha) - \sigma(\alpha)\gamma$, सभी $\alpha \in \mathcal{A}$ के लिए। यदि D इनर नहीं है, तो इसे आउटर कहा जाता है। जब $\sigma = \tau$ है, तो एक (σ, τ) -डेरिवेशन, इनर (σ, τ) -डेरिवेशन, और आउटर (σ, τ) -डेरिवेशन को क्रमशः σ -डेरिवेशन, इनर σ -डेरिवेशन, और आउटर σ -डेरिवेशन कहा जाता है। जब $\sigma = \tau = \text{id}_{\mathcal{A}}$ तो एक (σ, τ) -डेरिवेशन सामान्य लीबनिज़ नियम को संतुष्ट करती है: $D(\alpha\beta) = D(\alpha)\beta + \alpha D(\beta)$, सभी $\alpha, \beta \in \mathcal{A}$ के लिए, और फिर एक (σ, τ) -डेरिवेशन, इनर (σ, τ) -डेरिवेशन, और आउटर (σ, τ) -डेरिवेशन को क्रमशः डेरिवेशन, इनर डेरिवेशन और आउटर डेरिवेशन कहा जाता है। समान रूप से, हम इन अवधारणाओं को एक मनमाने रिंग R के ट्विस्टेड डेरिवेशनस के लिए परिभाषित करते हैं। लेकिन इस मामले में, हम R -लीनियरिटी की स्थिति पर विचार नहीं करते हैं। यहाँ, D को केवल अद्वितीय होना चाहिए और ट्विस्टेड लीबनिज़ नियम को संतुष्ट करना चाहिए।

अध्याय 1 में, हम शोध प्रबंध में किए गए कार्य का परिचय देते हैं। हम थीसिस की ABT संरचना पर चर्चा करते हैं। "A" का अर्थ है "और", जो साहित्य में पहले से ही ज्ञात है। यह अध्याय 1 की धारा 1.1, 1.2, 1.3 और 1.4 में पाया जा सकता है। खंड 1.1 में, हम इस विषय को उठाने के पीछे की प्रेरणा पर चर्चा करते हैं। खंड 1.2 विकसित सिद्धांत को समझने के लिए आवश्यक बुनियादी परिभाषाओं पर चर्चा करता है। खंड 1.3 में ग्रुप रिंग्स में ट्विस्टेड डेरिवेशनस पर साहित्य पर चर्चा की गई है। धारा 1.4 में, हम अलजेब्राइक नंबर फ़ील्ड्स में ट्विस्टेड डेरिवेशनस पर उपलब्ध साहित्य पर चर्चा करते हैं। "B" का अर्थ है "लेकिन", जो मौजूदा साहित्य में उन अंतरालों को दर्शाता है जिन्हें भरने की आवश्यकता है और अज्ञात जिन्हें खोजने की आवश्यकता है। अंत में, "T" का अर्थ है "इसलिए", जो दर्शाता है कि हम आगे क्या करने का लक्ष्य रखते हैं। "B" और "T" की चर्चा अध्याय 1 की धारा 1.5 में की गई है, जिसमें हम इस थीसिस में काम की गई शोध समस्याओं पर चर्चा करते हैं।

अध्याय 2 में, हम थीसिस के शेष अध्यायों में आवश्यक प्रारंभिक बातों पर चर्चा करते हैं। धारा 2.1 में, हम ट्विस्टेड डेरिवेशनस पर कुछ आवश्यक लेम्मा का उल्लेख करते हैं। खंड 2.2 में, हम ग्रुप सिद्धांत से कुछ ज्ञात परिणाम देते हैं जैसे कि कुछ महत्वपूर्ण ग्रुप्स के कंज्यूगेसी क्लासेज पर परिणाम (उदाहरण के लिए, डायहेड्रल ग्रुप D_{2n} , डाइसाइक्लिक ग्रुप T_{4n} , अर्ध-डायहेड्रल ग्रुप SD_{8n} और V_{8n}), फ्री ग्रुप्स की यूनिवर्सल प्रॉपर्टी, डबली-ट्विस्टेड कंज्यूगेसी क्लासेज और उन पर कुछ परिणाम, D_{2m} से D_{2n} तक ग्रुप होमोमोर्फिजम्स। सेक्शन 2.3 में, हम ग्रुप रिंग्स के सिद्धांत से कुछ प्रसिद्ध परिभाषाओं और परिणामों को बताते हैं, जैसे कि ऑगमेंटेशन आइडियल, मास्चके थ्योरम, वेडरबर्न-आर्टिन थ्योरम, क्लास सम, ग्रुप रिंग मैट्रिक्सज की अवधारणा, और इन ग्रुप मैट्रिक्सज से संबंधित विभिन्न रोमांचक और मूल्यवान परिणाम। खंड 2.4 में, हम कम्यूटेटिव रिंग्स पर लीनियर अलजेब्रा के सिद्धांत से कुछ परिणामों पर चर्चा करते हैं। खंड 2.5 में, हम अलजेब्राइक नंबर सिद्धांत से बुनियादी परिभाषाओं और परिणामों को बताते हैं। खंड 2.6 में, हम कोडिंग सिद्धांत से कुछ बुनियादी परिभाषाओं और ग्रुप रिंग्स से जीरो डिविज़र और यूनिट-व्युत्पन्न कोड पर कुछ ज्ञात परिणामों पर चर्चा करते हैं।

अध्याय 3 में, हम कुछ प्रभावशाली ग्रुप्स, अर्थात् डाइहेड्रल (D_{2n}), डाइसाइक्लिक (T_{4n}) और सेमी-डाइहेड्रल (SD_{8n}) के ग्रुप अलजेब्रा की डेरिवेशनस का अध्ययन करते हैं। सबसे पहले, हम स्पष्ट रूप से एक फाइनाइट ग्रुप G के ग्रुप अलजेब्रा के सभी इनर डेरिवेशनस को एक मनमाने फील्ड \mathbb{F} पर वर्गीकृत करते हैं। फिर हम ग्रुप अलजेब्रा $\mathbb{F}D_{2n}$, $\mathbb{F}T_{4n}$ और $\mathbb{F}(SD_{8n})$ के सभी \mathbb{F} -डेरिवेशनस को वर्गीकृत करते हैं इन डेरिवेशन अलजेब्रा का डायमेंशन और एक स्पष्ट बेसिस देकर जब \mathbb{F} कैरेक्टरिस्टिक 0 या एक ओड रेशनल प्राइम p की फील्ड है। हम स्पष्ट रूप से एक मनमाने फील्ड पर इन ग्रुप अलजेब्रा की सभी इनर डेरिवेशनस का वर्णन करते हैं। अंत में, हम उपरोक्त सभी ग्रुप अलजेब्रा डेरिवेशनस को तब वर्गीकृत करते हैं जब \mathbb{F} एक प्राइम फील्ड का अलजेब्राइक विस्तार होता है। विशेष रूप से, जब \mathbb{F} एक प्राइम फील्ड का अलजेब्राइक विस्तार होता है, तो हम एक पर्याप्त शर्त देते हैं जिसके तहत इन ग्रुप अलजेब्रा में एक गैर-तुच्छ आउटर डेरिवेशन होती है, इस प्रकार इन ग्रुप अलजेब्रा में डेरिवेशन समस्या का उत्तर देते हैं।

अध्याय 4 में, हम V_{8n} , जो $8n$ क्रम (n एक पॉजिटिव इन्टिजर) का एक ग्रुप है प्रेजेंटेशन $\langle a, b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b \rangle$ के साथ, के ग्रुप अलजेब्रा $\mathbb{F}V_{8n}$ के इनर और आउटर डेरिवेशन्स का अध्ययन करते हैं, एक कैरेक्टरिस्टिक 0 या एक ओड रेशनल प्राइम p फील्ड \mathbb{F} पर। हम स्पष्ट रूप से ग्रुप अलजेब्रा $\mathbb{F}V_{8n}$ के सभी इनर और आउटर डेरिवेशन्स को वर्गीकृत करते हैं। सबसे पहले, हम $\mathbb{F}V_{8n}$ के सभी \mathbb{F} -डेरिवेशन्स से मिलकर डेरिवेशन अलजेब्रा का डायमेंशन और एक स्पष्ट बेसिस देकर $\mathbb{F}V_{8n}$ के सभी \mathbb{F} -डेरिवेशन्स को स्पष्ट रूप से वर्गीकृत करते हैं। नतीजतन, हम $\mathbb{F}V_{8n}$ के सभी इनर और आउटर डेरिवेशन्स को वर्गीकृत करते हैं जब \mathbb{F} एक प्राइम फील्ड का अलजेब्राइक विस्तार होता है। इस प्रकार हम यह स्थापित करते हैं कि $\mathbb{F}V_{8n}$ की सभी डेरिवेशन्स इनर होती हैं जब \mathbb{F} की कैरेक्टरिस्टिक 0 या p होती है, p और n अपेक्षाकृत प्राइम होते हैं, और गैर-शून्य आउटर डेरिवेशन्स केवल उस स्थिति में मौजूद होती हैं जब \mathbb{F} की कैरेक्टरिस्टिक p होती है, p और n अपेक्षाकृत प्राइम नहीं होते हैं।

अध्याय 5 में, हम ग्रुप के जनरेटर और रिलेटर के संदर्भ में ग्रुप रिंग्स के (σ, τ) -डेरिवेशन्स का अध्ययन करते हैं। लियो क्रीडन और कीरन ह्यूजेस [25] ने ग्रुप G के जनरेटर और रिलेटर के संदर्भ में ग्रुप रिंग RG (R एक कम्यूटेटिव यूनिटल रिंग) की डेरिवेशन्स का अध्ययन किया। इस अध्याय में, हम (σ, τ) -डेरिवेशन्स के लिए ऐसा करते हैं। हम एक आवश्यक और पर्याप्त शर्त विकसित करते हैं कि एक मैप $f: X \rightarrow RG$ को RG के एक (σ, τ) -डेरिवेशन D तक विशिष्ट रूप से विस्तारित किया जा सकता है, जहां R यूनिटी के साथ एक कम्यूटेटिव रिंग है, G एक प्रस्तुति $\langle X|Y \rangle$ (X जनरेटरों का सेट और Y रिलेटरों का सेट) और $(\sigma, \tau) RG$ के एंडोमोर्फिज्म की एक जोड़ी है जो G के ग्रुप होमोमोर्फिज्म के R -लीनियर एक्सटेंशन हैं। इसके अलावा, हम एक फाइनाइट ग्रुप G के ग्रुप अलजेब्रा $\mathbb{F}G$ के सभी इनर (σ, τ) -डेरिवेशन्स को एक मनमाने फील्ड \mathbb{F} पर डायमेंशन और $\mathbb{F}G$ के सभी इनर (σ, τ) -डेरिवेशन्स से युक्त संबंधित \mathbb{F} -वेक्टर स्पेस के स्पष्ट बेसिस के संदर्भ में वर्गीकृत करते हैं। हम यह भी साबित करते हैं कि यदि R एक यूनिटल रिंग है और G एक ग्रुप है जिसका क्रम R में इन्वर्टिबल है, तो RG का प्रत्येक (σ, τ) -डेरिवेशन इनर है। हम अंत में ऊपर प्राप्त परिणामों को किसी भी कैरेक्टरिस्टिक के एक मनमाने फील्ड \mathbb{F} पर डायहेड्रल ग्रुप अलजेब्रा $\mathbb{F}D_{2n}$ ($D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, $n \geq 3$) के सभी इनर और आउटर σ -डेरिवेशन्स को वर्गीकृत करने के लिए लागू करते हैं, इस प्रकार डायहेड्रल ग्रुप अलजेब्रा में ट्विस्टेड डेरिवेशन समस्या का उत्तर देते हैं।

अध्याय 6 में, हम एक फाइनाइट साइक्लिक ग्रुप G के ग्रुप रिंग RG के इनर और आउटर ट्विस्टेड या (σ, τ) -डेरिवेशन्स का अध्ययन एक कम्यूटेटिव यूनिटल रिंग R पर करते हैं। मान लीजिए कि R यूनिटी के साथ एक कम्यूटेटिव रिंग है, G एक फाइनाइट साइक्लिक ग्रुप है, और (σ, τ) ग्रुप अलजेब्रा RG के R -अलजेब्रा एंडोमोर्फिज्म का एक जोड़ा है, जो G के ग्रुप एंडोमोर्फिज्म के R -लीनियर विस्तार हैं। हम ग्रुप रिंग RG के (σ, τ) -डेरिवेशन्स से संबंधित दो लक्षण वर्णन देते हैं। सबसे पहले, हम एक R -लीनियर मैप $D: RG \rightarrow RG$, $D(1) = 0$ के साथ, को एक (σ, τ) -डेरिवेशन होने के लिए एक आवश्यक और पर्याप्त शर्त प्रदान करते हैं। दूसरा, हम RG के एक (σ, τ) -डेरिवेशन को इनर होने के लिए एक आवश्यक और पर्याप्त स्थिति विकसित करते हैं। हम उदाहरणों की मदद से अपने प्रमेयों को भी स्पष्ट करते हैं। इन दो लक्षण वर्णनों के परिणामस्वरूप, हम RG के लिए प्रसिद्ध ट्विस्टेड डेरिवेशन समस्या का उत्तर देते हैं: किन परिस्थितियों में RG के सभी $(\sigma,$

τ -डेरिवेशन्स इनर हैं? या आउटर ट्विस्टेड डेरिवेशन्स का स्थान तुच्छ है? अधिक सटीक रूप से, हम एक पर्याप्त शर्त देते हैं जिसके तहत RG के सभी (σ, τ) -डेरिवेशन्स इनर और एक पर्याप्त स्थिति है जिसके तहत RG में गैर-तुच्छ आउटर (σ, τ) -डेरिवेशन्स हैं। हमारा परिणाम गैर-तुच्छ आउटर (σ, τ) -डेरिवेशन्स के कई उदाहरण उत्पन्न करने में मदद करता है।

अध्याय 7 में, हम नंबर रिंग्स के (σ, τ) -डेरिवेशन्स का अध्ययन उन्हें कम्प्यूटिव यूनिटल \mathbb{Z} -अलजेब्रा के रूप में मानते हुए करते हैं। हम एक क्वाड्रटिक नंबर फ़िल्ड के अलजेब्राइक इन्टिजर्स के रिंग के सभी (σ, τ) -डेरिवेशन्स और इनर (σ, τ) -डेरिवेशन्स को चिह्नित करके शुरू करते हैं। फिर हम एक p^{th} -साइक्लोटोमिक नंबर फ़िल्ड $\mathbb{Q}(\zeta)$ (p ओड रेशनल प्राइम और ζ यूनिटी का एक प्रिमिटिव p^{th} -रूट) के अलजेब्राइक इन्टिजर्स $\mathbb{Z}[\zeta]$ के रिंग की सभी (σ, τ) -डेरिवेशन्स को चिह्नित करते हैं। हम (SAGE और MATLAB का उपयोग करते हुए) $\mathbb{Z}[\zeta]$ पर एक (σ, τ) -डेरिवेशन D के इनर होने के लिए एक "यदि और केवल यदि" शर्त का भी अनुमान लगाते हैं। हम आगे बाय-क्वाड्रटिक नंबर रिंग $\mathbb{Z}[\sqrt{m}, \sqrt{n}]$ (m, n अलग-अलग स्क्वायर-फ्री रेशनल इन्टिजर्स) के सभी (σ, τ) -डेरिवेशन्स और इनर (σ, τ) -डेरिवेशन्स को चिह्नित करते हैं। उपरोक्त प्रत्येक मामले में, हम नंबर रिंग के सभी (σ, τ) -डेरिवेशन्स से मिलकर डेरिवेशन अलजेब्रा की रैंक और एक स्पष्ट बेसिस भी निर्धारित करते हैं।

अध्याय 8 में, हम अलजेब्राइक नंबर फ़िल्ड्स में ट्विस्टेड डेरिवेशन्स का अध्ययन करते हैं। मान लीजिए कि A , यूनिटी के साथ एक कम्प्यूटिव रिंग है, $B = A[\theta]$ (A पर कुछ $\theta \in B$ इंटिग्रल के लिए) कोशेंट फ़िल्ड \mathbb{K} के साथ एक इंटिग्रल डोमेन है, और \mathbb{E}, \mathbb{K} पर θ का न्यूनतम स्प्लिटिंग फ़िल्ड है। मान लीजिए कि $\sigma, \tau: B \rightarrow \mathbb{E}$ रिंग होमोमोर्फिज्म हैं जो A को एलिमेंट-वाइज फिक्स करते हैं। हम साबित करते हैं कि एक A -लीनियर मैप $D: B \rightarrow \mathbb{E}$, $D(1) = 0$ के साथ, जो कुछ शर्तों को संतुष्ट करता है, एक (σ, τ) -डेरिवेशन है। नतीजतन, हम अलजेब्राइक नंबर फ़िल्ड्स और उनके अलजेब्राइक इन्टिजर्स के रिंग में ट्विस्टेड डेरिवेशन्स पर परिणाम प्राप्त करते हैं। साइक्लोटोमिक नंबर फ़िल्ड $\mathbb{K} = \mathbb{Q}(\zeta)$ (ζ एक n^{th} यूनिटी का प्रिमिटिव रूट) के अलजेब्राइक इन्टिजर्स के रिंग $O_K = \mathbb{Z}[\zeta]$ के लिए और O_K के दो अलग-अलग \mathbb{Z} -अलजेब्रा एंडोमोर्फिज्म का एक युग्म (σ, τ) के लिए, हम एक (σ, τ) -डेरिवेशन $D: O_K \rightarrow O_K$ के इनर होने के लिए एक आवश्यक और पर्याप्त शर्त का अनुमान लगाते हैं। यह n के दो अलग-अलग रूपों के लिए किया जाता है: (i) $n = 2^r p$ ($r \in \mathbb{N}$ और p एक ओड रेशनल प्राइम) और (ii) $n = p^k$ ($k \in \mathbb{N}$ और p कोई रेशनल प्राइम)। हम n के उपरोक्त दो रूपों के लिए गैर-शून्य आउटर ट्विस्टेड डेरिवेशन्स के अस्तित्व और गैर-अस्तित्व का भी अनुमान लगाते हैं, इस प्रकार ट्विस्टेड डेरिवेशन्स समस्या का उत्तर देते हैं।

अध्याय 9 में, हम बीजगणितीय कोडिंग सिद्धांत में 3 से 8 तक पिछले छह अध्यायों में किए गए कार्य के अनुप्रयोगों का अध्ययन करते हैं। हम अच्छे पैरामीटर कोड का निर्माण करते हैं, जिनमें से कुछ में अतिरिक्त गुण होते हैं, जैसे कि ऑप्टीमल, सेल्फ-डुअल, स्व-ऑर्थोगोनल और एलसीडी। हम "डेरिवेशन इमेज-व्युत्पन्न कोड की" की धारणा देते हैं। इसके अलावा, हम दिखाते हैं कि डेरिवेशन्स जीरो डिवाइज़र और यूनिट-व्युत्पन्न कोड के रूप में होती हैं।

अंत में, अध्याय 10 में, हम थीसिस के मुख्य योगदानों पर प्रकाश डालते हुए सभी थीसिस निष्कर्षों को समाप्त करते हैं। हम इस थीसिस में किए गए काम से उत्पन्न होने वाले भविष्य के कई शोध निर्देश भी देते हैं।

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List of Symbols

For a commutative unital ring R , and R -algebras \mathcal{A} , \mathcal{B} with $\mathcal{A} \subseteq \mathcal{B}$, we use the following notations:

Symbol	Meaning
\mathbb{N}	the set of natural numbers
\mathbb{N}_0	$\mathbb{N} \cup \{0\}$
\mathbb{Z}	the ring of integers
\mathbb{Q}	the field of rational numbers
\mathbb{R}	the field of real numbers
\mathbb{C}	the field of complex numbers
S_k	$\{(i, j) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid i + j = k\}$ for $k \in \mathbb{N}_0$
$\gcd(r, s)$	the greatest common divisor of any two non-zero integers r and s
$\phi(n)$	the Euler's phi function of the positive integer n
$\Phi_n(x)$	the n^{th} cyclotomic polynomial
$ S $	the cardinality of a set S
$ G $	the order of a group G
C_n	a cyclic group of order n
$ g $	the order of an element g in a group G
$[G : H]$	the index of a subgroup H of a group G
$Q < \infty$	to indicate that some quantity Q is finite
$Z(G)$	the center of a group G
$Z(R')$	the center of a ring R'
$Z(R'G)$	the center of the group ring $R'G$ of a group G over a ring R'
$O_{\mathbb{K}}$	the ring of algebraic integers of a number field \mathbb{K}

$[\mathbb{K}, \mathbb{F}]$	the degree of the field extension \mathbb{K} of a field \mathbb{F}
\mathbb{F}_{p^m}	finite field of order p^m (p prime, $m \in \mathbb{N}$)
$\text{char}(\mathbb{F})$	the characteristic of a field \mathbb{F}
$\text{id}_{R'}$	the identity endomorphism on a ring R'
$M_{m \times n}(R')$	the ring of $m \times n$ matrices ($m, n \in \mathbb{N}$) with entries from a ring R'
$M_n(R')$	the ring of $n \times n$ matrices ($n \in \mathbb{N}$) with entries from a ring R'
I_n	the $n \times n$ identity matrix over a commutative unital ring
A^T	the transpose of a matrix A
$\text{Adj}(A)$	the adjoint of a matrix A
$\det(A)$	the determinant of a matrix A
$\text{Der}_S(R')$	the set of all S -derivations ($D : R' \rightarrow R'$) of a ring R'
$\text{Der}(R')$	the set of all derivations ($D : R' \rightarrow R'$) of a ring R'
$\text{Der}_{\text{inn}}(R')$	the set of all inner derivations ($D : R' \rightarrow R'$) of a ring R'
$\mathcal{D}_{(\sigma, \tau)}(\mathcal{A}, \mathcal{B})$	the set of all (σ, τ) -derivations $D : \mathcal{A} \rightarrow \mathcal{B}$
$\text{Inn}_{(\sigma, \tau)}(\mathcal{A}, \mathcal{B})$	the set of all inner (σ, τ) -derivations $D : \mathcal{A} \rightarrow \mathcal{B}$
$\mathcal{D}_{(\sigma, \tau)}(\mathcal{A})$	the set of all (σ, τ) -derivations $D : \mathcal{A} \rightarrow \mathcal{A}$, that is, $\mathcal{B} = \mathcal{A}$
$\text{Inn}_{(\sigma, \tau)}(\mathcal{A})$	the set of all inner (σ, τ) -derivations $D : \mathcal{A} \rightarrow \mathcal{A}$, that is, $\mathcal{B} = \mathcal{A}$
$\mathcal{D}_\sigma(\mathcal{A})$	the set of all σ -derivations $D : \mathcal{A} \rightarrow \mathcal{A}$, that is, $\tau = \sigma$
$\text{Inn}_\sigma(\mathcal{A})$	the set of all inner σ -derivations $D : \mathcal{A} \rightarrow \mathcal{A}$, that is, $\tau = \sigma$
$\mathcal{D}(\mathcal{A})$	the set of all ordinary derivations $D : \mathcal{A} \rightarrow \mathcal{A}$, that is, $\sigma, \tau = \text{id}_{\mathcal{A}}$
$\text{Inn}(\mathcal{A})$	the set of all ordinary inner derivations $D : \mathcal{A} \rightarrow \mathcal{A}$, that is, $\sigma, \tau = \text{id}_{\mathcal{A}}$