UNITS, IDEMPOTENTS, AND IDEALS IN CERTAIN ALGEBRAS OVER COMMUTATIVE RINGS

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UNITS, IDEMPOTENTS, AND IDEALS IN CERTAIN ALGEBRAS OVER COMMUTATIVE RINGS

by

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **Units, Idempotents, and Ideals in Certain Algebras over Commutative Rings** submitted by **Ms. Meenu Khatkar** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
October, 2018

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Abstract

Idempotents and units in rings play a very important role in the study of rings. Several classes of elements are defined using idempotents and units, for example, clean elements, strongly clean elements, unit regular elements, Lie regular elements,..., etc. Due to their importance, the idempotents and units generated interest among several researchers and efforts have been made to compute idempotents and unit groups of various rings. Kanwar, Leroy, and Matczuk have discussed idempotents of polynomial rings and Laurent polynomial rings. Not much, however, is known in the case of polynomial rings over matrix rings (equivalently, matrix rings over polynomial rings). In this thesis, we give the idempotents of $M_2(\mathbb{Z}_p[x])$ for any prime p, of $M_2(\mathbb{Z}_{2p}[x])$ for any odd prime p, and of $M_2(\mathbb{Z}_{3p}[x])$ for any prime p greater than 3. We also prove that for any ring R, the unit group of $M_n(R[x])$ is not solvable. We also obtain the form of units in matrix rings $M_2(\mathbb{Z}_2[x])$ and $M_2(\mathbb{Z}_3[x])$.

It is worth observing that both matrix rings and polynomial rings are algebras over the field (ring) associated with them. In 2005, Abrams and Aranda Pino observed that matrix rings and Laurent polynomial rings over a field K can be realized as algebras of certain graphs. In fact, for any (row-finite) directed graph E and a field K, they introduced the idea of Leavitt path algebra $L_K(E)$. Ara, Moreno, and Pardo also independently introduced the concept of Leavitt path algebras. These Leavitt path algebras have been of interest to algebraists as well as analysts due to their connections with various algebraic structures. Several generalizations of these algebras have also been studied in the last decade. On one hand, Abrams and Aranda Pino generalized the concept to arbitrary graphs and on the other, Tomforde considered these algebras with the coefficients from a commutative unital ring instead of a field. In this thesis, we continue the study of Leavitt path algebras with coefficients from a commutative unital ring. Specifically, we study the basic ideals and basic left (right) ideals of Leavitt path algebras $L_R(E)$ over a commutative unital ring R. We show that for a finite acyclic graph E and a commutative unital ring R, the Leavitt path algebra $L_R(E)$ is a direct sum of minimal basic ideals. We also discuss uniqueness theorems for Leavitt path algebras over commutative unital rings. We also prove that the Leavitt path algebra $L_R(E)$ over a commutative unital ring R is non-degenerate if and only if R has no non-zero nilpotent elements, that is, R is a (commutative) semiprime ring. Further, we give conditions under which a basic left (right) ideal of Leavitt path algebra $L_R(E)$ generated by a vertex is a minimal basic left (right) ideal. We also show that if R has no non-zero nilpotent elements then every minimal basic left ideal $L_R(E)x$ of the Leavitt path algebra $L_R(E)$ contains a vertex.

सार

आइडेम्पोटेंट्स और युनिट्स, रिंग्स के अध्ययन में बहुत ही महत्वपूर्ण भूमिका निभाते हैं। आइडेम्पोटेंट्स और युनिट्स का उपयोग करके, एलिमेंट्स के कई अन्य वर्गों को परिभाषित किया जा सकता है, जैसे-क्लीन एलिमेंट्स, स्ट्रोंगली क्लीन एलिमेंट्स, यूनिट रेगुलर एलिमेंट्स, ली रेगुलर एलिमेंट्स... इत्यादि। अपने महत्व के कारण, आइडेम्पोटेंट्स और युनिट्स ने अनेक शोधकर्ताओं में रुचि उत्पन्न की है एवं बहुत सी रिंग्स के आइडेम्पोटेंट्स और यूनिट ग्रुप की गणना करने का प्रयास किया है। कंवर, लेरोय और मैत्जुक ने पोलीनोमियल रिंग्स एवं लोरेंट पोलीनोमियल रिंग्स के आइडेम्पोटेंट्स के विषय में चर्चा की है। यद्यपि, मैट्रिक्स रिंग्स के ऊपर पोलीनोमियल रिंग्स (समकक्ष, पोलीनोमियल रिंग्स के ऊपर मैट्रिक्स रिंग्स) के विषय में ज्यादा ज्ञात नहीं है। इस शोधग्रंथ में हमने किसी परिमेय संख्या p के लिए $M_2(Z_p[x])$ के, किसी विषम परिमेय संख्या p के लिए $M_2(Z_{2p}[x])$ के, और 3 से बड़े किसी परिमेय संख्या p के लिए $M_2(Z_{3p}[x])$ के आइडेम्पोटेंट्स ज्ञात किए हैं। हमने यह भी प्रमाणित किया है कि किसी रिंग R के लिए $M_n(R[x])$ का यूनिट ग्रुप सोल्वेबल नहीं होगा। हमने मैट्रिक्स रिंग्स $M_2(Z_2[x])$ और $M_2(Z_3[x])$ के युनिट्स के प्रकार भी ज्ञात किए हैं।

यह देखने योग्य है कि मैट्रिक्स रिंग्स और पोलीनोमियल रिंग्स दोनों ही उनके साथ जुड़ी हुई फील्ड (रिंग) के ऊपर एल्ज़ेब्रा हैं। 2005 में अब्राम और अरांडा पीनो ने यह पाया कि, किसी फील्ड K के ऊपर मैट्रिक्स रिंग्स और लोरेंट पोलीनोमियल रिंग्स को, किसी ग्राफ पर एल्ज़ेब्रा के रूप में समझा जा सकता है । वास्तव में, किसी भी रो-फाइनाइट डाइरेक्टेड ग्राफ E और फील्ड K के लिए, उन्होने लिविट पाथ एल्ज़ेब्रा $L_K[E]$ की अवधारणा को प्रस्तुत किया। एरा, मोरेनो, एवं पारडो ने भी लिविट पाथ एल्ज़ेब्रा के विचार को स्वतंत्र रूप से प्रस्तुत किया। ये लिविट पाथ एल्ज़ेब्राज विभिन्न एल्ज़ेब्राईक स्ट्रक्चर्स के साथ संबन्धित होने के कारण एल्ज़ेब्राइस्ट्स के साथ-साथ एनालिस्ट्स के लिए भी रूचि का विषय रहे हैं। पिछले दशक में इन एल्ज़ेब्राज़ के कई समीकरणों का अध्ययन किया गया है। एक तरफ अब्राम और पीनो ने इस विषय को आर्बिट्रेरी ग्राफ्स पर सामान्यीक्रत किया और दूसरी तरफ टोमफोर्ड ने फील्ड की जगह कम्युटेटिव यूनिटल रिंग्स से कोफिसीऐंट्स लेकर इन एल्ज़ेब्राज़ को माना। इस शोधग्रंथ में हम कम्युटेटिव यूनिटल रिंग्स से कोफिसीऐंट्स लेकर लिविट पाथ एल्ज़ेब्रा का आगे अध्ययन करते हैं। विशेष रूप से हमने कम्युटेटिव यूनिटल रिंग R के ऊपर लिविट पाथ

एल्ज़ेब्रा $L_R[E]$ के बेसिक आइडियल्स और बेसिक लेफ्ट (राइट) आइडियल्स का अध्य्यन किया है। हमने इसमे दिखाया है कि किसी फाइनाइट एसाइक्लिक ग्राफ E और कम्युटेटिव यूनिटल रिंग R के लिए लिविट पाथ एल्ज़ेब्रा $L_R[E]$ मिनिमल बेसिक आइडियल्स का डायरेक्ट सम है। हमने कम्युटेटिव यूनिटल रिंग्स के ऊपर लिविट पाथ एल्ज़ेब्राज़ के लिए यूनिकनेस थ्योरम्स की भी चर्चा की है। हमने यहाँ यह भी प्रमाणित किया है कि कम्युटेटिव यूनिटल रिंग R के ऊपर लिविट पाथ एल्ज़ेब्रा $L_R[E]$ नॉन-डिजेनेरेट होगा यदि और केवल यदि R में कोई नॉन-ज़ीरो निल्पोटेंट एलिमेंट ना हो, अर्थात, R एक कम्युटेटिव सेमीप्राइम रिंग है। हमने आगे वह परिस्थितियाँ भी दी हैं जिनमें एक वर्टेक्स से जेनेरेट हुए लिविट पाथ एल्ज़ेब्रा का बेसिक लेफ्ट (राइट) आइडियल एक मिनिमल बेसिक लेफ्ट (राइट) आइडियल है। इसके अतिरिक्त हमने यह भी दर्शाया है की अगर R में कोई नॉन-ज़ीरो निल्पोटेंट एलिमेंट नहीं है तब लिविट पाथ एल्ज़ेब्रा $L_R[E]$ के प्रत्येक मिनिमल बेसिक लेफ्ट आइडियल $L_R[E]x$ के अंदर एक वर्टेक्स होगा।

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Symbol	Meaning
$=$	equal to
$\neq$	not equal to
$\forall$	for all
$\in$	belongs to
$\notin$	does not belong to
$\subseteq$	subset or equal
$\subsetneq$	proper subset
$\cup$, $\cap$	union, intersection
$\cong$	isomorphic to
$\equiv$	congruent to
$\oplus$	direct sum
$X \setminus E$	the complement of E in X
$\emptyset$	empty set
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers

$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the real line
$\mathbb{Z}_n$	the group of integers modulo n
$R[x]$	the polynomial ring over R
$M_n(R)$	the $n \times n$ matrix ring over R
$E(R)$	the set of idempotents in R
$\mathcal{U}(R)$	the set of units of R
(g_1, g_2)	the commutator $g_1^{-1}g_2^{-1}g_1g_2$ of group elements g_1 and g_2
$\delta(G)$	the derived subgroup of G
$\delta^n(G)$	the n^{th} derived subgroup of G
$\phi(n)$	Euler's phi function
$GL(n, R)$	the general linear group
$SL(n, R)$	the special linear group
$\gcd(x, y)$	the greatest common divisor of x and y