

MATHEMATICAL MODELS FOR OXYGEN AND CARBON DIOXIDE TRANSPORT IN THE PULMONARY AND SYSTEMIC CAPILLARIES

by

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(BALBIR SINGH)

CERTIFICATE

This is to certify that the thesis entitled "MATHEMATICAL MODELS FOR OXYGEN AND CARBON DIOXIDE TRANSPORT IN THE PULMONARY AND SYSTEMIC CAPILLARIES", being submitted by BALBIR SINGH for the award of the DEGREE OF DOCTOR OF PHILOSOPHY is a record of the original bonafide research work carried out by him. He has worked under our guidance and supervision and has fulfilled the requirements for the submission of this thesis. The results presented in this thesis have not been submitted in part or full to any other University or Institute for award of any degree/ diploma.



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ABSTRACT

The studies on the transport of gases in the pulmonary and systemic circulations are of physiological and pathological interests. The process of respiration essentially involves an uptake of oxygen (O_2) from the atmosphere and disposal of carbon dioxide (CO_2). In the present thesis, the studies on the transport of O_2 and CO_2 in the blood flowing through the pulmonary capillaries are concerned with the development of mathematical models for the process of blood oxygenation in the lungs. In the systemic circulation, the mathematical models are developed for the transport of O_2 and CO_2 to understand the reasons of discomfort felt by the divers in deep sea environment.

A mathematical model for the process of gas exchange leading to the oxygenation of blood in pulmonary capillary is developed. The model takes into consideration the transport mechanisms of molecular diffusion, convection and the facilitated diffusion. The nth order one step kinetics of O_2 uptake by haemoglobin is incorporated in the model. The formulation leads to a system of coupled nonlinear parabolic partial differential equations. In the first stage of the analysis, the equilibrium solution corresponding to arterial blood is obtained in the closed form. The results computed from our model are in good agreement with those based on experimental data. In the second stage of the analysis, the rate at which blood gets oxygenated in the capillary is obtained by setting up an appropriate eigenvalue problem. The analysis shows that the dissolved O_2 takes the longest in reaching equilibration whereas CO_2 is the fastest to attain equilibration - a result which may explain certain borderline respiratory disorders. It is shown that how the process of oxygenation suffers at high altitude and in muscular exercise.

The effect of entrance conditions on the rate of oxygenation has been analyzed by solving the nonlinear governing equations of the model numerically by a four point implicit scheme. A fixed point iterative technique is used to deal with the nonlinearities in the model. It is shown that the blood is required to traverse a larger distance in the capillary before getting fully oxygenated with nth order one step kinetics in comparison to the first order one step kinetics.

A mathematical model is proposed for analysing the transport of O_2 in the systemic capillary and the surrounding tissue in hyperbaric environment. The model takes into account the molecular diffusion in both axial and radial directions in the capillary and tissue, the convective effect of the blood, the saturation of haemoglobin with O_2 in the blood and the first order metabolic consumption rate in the tissue. The formulation gives rise to a nonlinear partial differential equation in the capillary region and a linear partial differential equation in tissue region. In the analysis, the nonlinear oxygen dissociation curve is represented by a linear function for simulating the conditions of hyperbaric environment. An analytical solution of the system of equations is obtained by the method of eigenfunction expansion. It is found that the significant radial diffusion of O_2 takes place in the initial part of the tissue close to the arterial end. The accumulation of O_2 in the tissue is found to be higher with the first order metabolism in comparison to the constant metabolic rate.

The simultaneous transport of O_2 and CO_2 is important because the presence of CO_2 affects the release of O_2 from haemoglobin to the tissue cells. Hence, a mathematical model is formulated to deal with the simultaneous transport of O_2 and CO_2 in systemic capillary and the surrounding tissue. The production of CO_2 inside the tissue is assumed to be proportional to the consumption of O_2 . The saturation

of blood with O_2 and CO_2 have been approximated by linear functions to simulate the hyperbaric conditions. Using this approximation, an analytical solution is obtained for the resulting system of coupled partial differential equations with the physiologically relevant boundary conditions. It is shown that very little CO_2 is transported radially. The location of the vulnerable point from the point of view of CO_2 accumulation is found to be the corner situated at the periphery of the tissue near the venous end of the capillary. It is surmised that one of the reasons of discomfort among divers could be excessive accumulation of CO_2 in the tissue.

The nonlinearity of oxygen dissociation curve plays an important role in the delivery of O_2 to the tissue. Hence, a mathematical model is formulated for the transport of O_2 in the systemic capillaries by incorporating the nonlinear effects of oxyhaemoglobin. The resulting convective-diffusion equation in the capillary is solved by the Finite Element Method. The computation is done for the plug flow profile. For solving mass transport equation, the velocity profiles are needed. For computing the flow, a finite element formulation is described using the creeping flow equations. Using the computed values of the velocities, the convective-diffusion equation is solved for the concentration distribution with the finite element method.

The above studies are based on the assumption that the blood is homogeneous. Since the size of suspended particles (red blood cells) is not negligible in comparison to that of the capillary, the blood can no longer be considered as a homogeneous fluid and hence, it is worthwhile to consider the blood flow as a two phase flow. A two layer model for the blood is described in order to examine the effect of the cell depleted layer on the transport of O_2 in the tissue.

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