SOME REFLECTIONS OF CONVERGENCE SPACES

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By

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Certificate

This is to certify that the thesis entitled 'Some Reflections of Convergence Spaces' which is being submitted by Vinod Kumar for the award of Doctor of Philosophy (Mathematics) to the Indian Institute of Technology, Delhi, is a record of bonafide research work.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results in this thesis have not been submitted to any othe University or Institute for the award of any degree or diploma.

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INTRODUCTION

Universal Construction, that is, 'embedding' an object in a suitably completed object that is universal, is of interest in any category. Universal constructions produce reflections and conversely, and characterization of reflective and coreflective subcategories of a category is considered a significant and interesting problem. In this context, Conv, the category of convergence spaces, it seems, has so far been investigated sparingly, although a lot of work has been done on convergence spaces (see references). Unfortunately some standard category theory tools usually helpful in characterising reflective subcategories of a category are not available for Conv in the light of Wyler's paper 'An unpleasant theorem for convergence spaces' [26]. Thus one is likely to think to figure out some important reflections of convergence spaces. A study of this nature is dealt with in this thesis.

A reflection of an object X is a suitably completed object, say $\bar{X}$, together with a morphism : $X \longrightarrow \bar{X}$, called reflection morphism, satisfying the universal property. In certain topological categories the reflection morphism turns out to be an embedding in some cases; such a reflection is called an embedding-reflection. For Hausdorff convergence spaces, we shall talk of embedding-reflections also. As for epireflections, in Conv an epi is onto and conversely. In H-Conv, the subcategory of Conv consisting of Hausdorff convergence spaces, the class of epi's is not interesting ([10, Proposition 1.2]). Seeing that a dense map is an epi and embeddings are usually dense, reflections with dense embeddings as reflection maps, to be called embedding-densereflections, may be studied in H-Conv.

First we investigate convergence spaces for embedding-densereflections. Since an extension of a convergence space, whenever it is a reflection, is, in fact, an embedding-densereflection, we try how various extensions, some of them known and some after constructing them, can be treated as reflections. In extensions, compactification is of vital importance. Compactification problems of convergence spaces have been studied by many, e.g., see [19], [20], [21] and [22], but not much has been said from the reflection view point. Since a Hausdorff compactification is a reflection iff it is universal, i.e., enjoys universal property, our problem is seemingly reduced to obtaining universal compactification.

Richardson [22] has constructed a compactification for every convergence space but that is not universal. If every convergence space is not expected to have the universal compactification, a class (possibly the largest) of convergence spaces having the universal compactification may be determined. Rao [20] and [21] has obtained necessary and sufficient conditions for a Hausdorff convergence space to have the largest compactification. We observe that the proof of the necessity part, which is given in detail in [21] only, is not sound.

In Chapter II, we obtain the largest class of convergence spaces having the universal compactification. This, besides giving the largest subcategory of H-Conv where compact Hausdorff embedding-densereflection exists, establishes the validity of Rao's result and determines the largest class of convergence spaces where the Richardson compactification can be treated as a reflection. In the sequel every Hausdorff convergence space is found to have atleast as many maximal compactifications as it has nonconvergent ultrafilters. Noting that the class of convergence spaces having the universal or largest compactification is very restrictive, one may look for some weak form compactness of which universal extension is possible for every Hausdorff convergence space. e-compactness appears to be a reasonably good choice. In topological spaces, the class of e-compactifiable spaces is not known ([7], [24]). For every Hausdorff convergence space, we construct an e-compactification that is universal and, in the language of category theory, gives an adjunction. Our e-compactification produces topological e-compactification for a particular class of topological spaces.

A pseudotopological compact Hausdorff convergence space being minimal Hausdorff, the pseudotopological modification of a

Hausdorff compactification of a Hausdorff convergence space is its minimal Hausdorff extension if the space is pseudo-topological. This fact coupled with the observation that a Hausdorff compactification is universal iff its topological modification is the universal minimal Hausdorff extension settles the question of the existence of minimal Hausdorff embedding-densereflection. Also, every maximal compactification of a pseudotopological Hausdorff convergence space gives a maximal minimal-Hausdorff extension of the space. To supplement the existence of maximal compactification (minimal-Hausdorff extension) a minimal compactification (minimal -Hausdorff extension) is also found to be existing.

Coming to ordinary reflections, in Chapter III, in the first place we describe Hausdorff and λ-Hausdorff reflections of a convergence space (cf. [9, Problem 5]). The existence of compact Hausdorff reflection is shown to be equivalent to that of compact Hausdorff embedding-densereflection (already discussed in Chapter II) resulting in coinciding the β-compactification of a Hausdorff convergence space, whenver it exists, with its universal compactification. This leads to defining β^{λ}- compactification which we are able to construct for every convergence space getting compact λ-Hausdorff reflection that makes compact convergence spaces epireflective in λ-Hausdorff convergence spaces. Also, the topological modification of the β^{λ}- compactification of a convergence space is the topological β-compactification of the convergence space and its topological modification. This further signifies the relevance of this new notion. Our construction of β^{λ} -compactification suggests a different and more explicit construction of the topological β-compactification for every convergence space (cf. [18, Theorem 7.1]).

Regarding minimal Hausdorff reflections we show that β-minimal Hausdorff extension of a pseudotopological Hausdorff convergence space is its minimal Hausdorff extension, and β^{λ}- minimal Hausdorff extension its topological β-compactification. This rules out the possibility of a new reflection. Minimal λ-Hausdorff convergence spaces and λ-minimal Hausdorff convergence spaces are found to be just topologically minimal Hausdorff topological spaces.

Now the result of Herrlich and Strecker [8] that H-Top, the category of Hausdorff topological spaces, itself is its only reflective subcategory that contains minimal Hausdorff topological spaces, implies that minimal λ-Hausdorff reflection of convergence spaces does not exist for any full subcategory of Conv. In Chapter IV, we obtain minimal λ-Hausdorff reflection of Hausdorff topological spaces in a 'different set up' that also settles a problem of Herrlich and Strecker [8].

In relation to coreflections of convergence spaces, we do not discuss much. We find two, namely, almost locally compact and locally compact coreflections of convergence spaces in Chapter V. Both the coreflections are shown to commute with the pseudotopological modification functor and finite products of convercoreflection gives the famous topological k-space coreflection. We also discuss some hereditary and productive properties of almost local compactness and local compactness.

Coming back to reflections, in Chapter VI, we study (topological) compact Hausdorff reflection of a topological space in regard to extremal disconnectedness, i.e., when it is extremally disconnected. We show that compact Hausdorff reflection is extremally disconnected if the topological space is extremally disconnected, but the converse, in contrast to the Stone-Čech compactification case, is not true. For the construction of compact Hausdorff reflection of an extremally disconnected topological space, we investigate open filters for a property which, we find, characterises extremal disconnectedness, and shows that the result of Exercise 12 E.6 (p. 83) of General Topology by S. Willard [25] does not hold for open filters.

First chapter contains definitions, notations and preliminaries that we use in the thesis; Chapters IV and VI are self contained in this regard.

Most of the results reported in Chapter II have appeared in Bull. Austral. Math. Soc., 16 (1977), 189 - 197 and Proc. AMS, 73 (1979), 256 - 262. The results of Chapters IV and VI have been accepted for publication under the titles 'A filter space functor' and 'Open filters and an e.d. extension' respectively in Topology and its Applications.

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