SEMI-CLASSICAL ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS IN LATTICES

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by

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Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

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Dedicated to my parents.

Certificate

This is to certify that the thesis entitled **Semi-classical Analysis of Partial Differential Equations in Lattices** submitted by **Mr. Abhilash Tushir** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

In this thesis, we investigate the semi-classical analysis of certain semi-discretized partial differential equations. These equations include the classical wave equation, the fractional Klein-Gordon equation, the heat equation with fractional Laplacian operator, the wave equation with Schrödinger operator, and the general time-fractional diffusion equation associated with Schrödinger operator. A semi-discretization method for solving PDEs discretizes one or more independent variables while leaving the others continuous. In our case, we keep the continuity of the time variable but make the space variable discrete by converting the continuous domain $\mathbb{R}^n$ to the discrete lattice $\hbar\mathbb{Z}^n$, where each point is a grid node. The space derivatives such as gradient, Laplacian, and fractional Laplacian are approximated by the finite difference operators. Here, the PDE is converted into a set of time-dependent ordinary differential equations (ODEs). This strategy is employed when the system's spatial behavior is of significant interest and the time domain is continuous or easier to analyze. The decision to employ semi-discretization is frequently decided by the nature of the problem, the computational resources available, and the specific properties of the PDE being solved. However, semi-discretization must be considered very carefully, and the chosen technique must align with the simulation targets and PDE characteristics.

As an introduction, Chapter 1 provides background information and the status of the study that will be covered in the upcoming chapters. We also present an overview of the thesis's organization and emphasize our essential contributions. In Chapter 2, we begin with the classical wave equation with smooth time-dependent coefficients and proceed to develop the well-posedness results. However, we also recover the well-posedness results

in the limit of the semi-classical parameter $\hbar \to 0$. In Chapter 3, our analysis is then extended to include the fractional Laplacian, and we take into consideration the fractional Klein-Gordon equation with a space-dependent bounded potential, and obtain the well-posedness, and recapture the Euclidean solution in the semi-classical limit $\hbar \to 0$. In Chapter 4, we then expand our analysis to the fractional heat equation, permitting the time-dependent coefficients to have irregularities and the initial Cauchy data to grow polynomially. We develop the well-posedness result for the case of regular coefficients in the space of discrete tempered distributions and see how the notion of a very weak solution adapts in such equations when coefficients have a distributional singularity. We also analyze how the very weak solutions can be approximated in the semi-classical limit $\hbar \to 0$. In Chapter 5, we investigate a particular class of wave equation for operators with a discrete non-negative spectrum, i.e., wave equation associated with the Schrödinger operator, and take into account the situation when potential blows up to infinity as $|k| \to \infty$. We first study the spectrum of the discrete Schrödinger operator and prove that it has a purely discrete spectrum when the potential blows up to infinity as $|k| \to \infty$. We analyze how the well-posedness results, in this case, vary with the classical wave equation, the fractional Klein-Gordon equation, and the fractional heat equation. In Chapter 6, we extend our investigation for the general Caputo-type time-fractional diffusion equation with the Schrödinger operator and consider the case when potential blows up to infinity as $|k| \to \infty$. At the end of the thesis, we would like to draw attention to a few research problems that have emerged throughout the process of writing the thesis.

सार

इस शोध प्रबंधन में, हम कुछ अर्ध-पृथकरण आंशिक अवकलन समीकरणों (पीडीई) के अर्ध-उत्कृष्टीय विश्लेषण की जांच करते हैं। इन समीकरणों में उत्कृष्टीय तरंग समीकरण, भिन्नात्मक क्लेन-गॉर्डन समीकरण, भिन्नात्मक लाप्लासियन संकारक के साथ तापीय समीकरण, श्रो-डिंगर संकारक के साथ तरंग समीकरण, और श्रो डिंगर संकारक से जुड़े सामान्य समय-भिन्नात्मक प्रसार समीकरण शामिल हैं। पीडीई को हल करने के लिए एक अर्ध-पृथकरण विधि एक या अधिक स्वतंत्र चर को अलग करती है जबकि अन्य को निरंतर छोड़ देती है। हमारे मामले में, हम समय चर की निरंतरता को बनाए रखते हैं लेकिन निरंतर प्रान्त $\mathbb{R}^n$ को असतत जाली $\hbar\mathbb{Z}^n$ में परिवर्तित करके समष्टि चर को असतत बनाते हैं, जहां प्रत्येक बिंदु एक जाल-संधि है। प्रवणता, लाप्लासियन और भिन्नात्मक लाप्लासियन जैसे समष्टि व्युत्पन्न परिमित अंतर संकारको द्वारा अनुमानित हैं। यहां, पीडीई को समय-निर्भर साधारण अवकलन समीकरणों (ओडीई) के एक समुचय में परिवर्तित किया जाता है। यह रणनीति तब नियोजित की जाती है जब तंत्र का स्थानिक व्यवहार महत्वपूर्ण रुचि का होता है और समय प्रान्त निरंतर या विश्लेषण करने में आसान होता है। अर्ध-पृथकरण को नियोजित करने का निर्णय अक्सर समस्या की प्रकृति, उपलब्ध सांगणात्मक संसाधनों और हल किए जा रहे पीडीई के विशिष्ट गुणों द्वारा तय किया जाता है। हालाँकि, अर्ध-पृथकरण पर बहुत सावधानी से विचार किया जाना चाहिए, और चुनी गई तकनीक को अनुकरणीय लक्ष्य और पीडीई विशेषताओं के साथ संरेखित करना चाहिए।

एक परिचय के रूप में, अध्याय 1 पृष्ठभूमि की जानकारी और अध्ययन की स्थिति प्रदान करता है जिसे आगामी अध्यायों में शामिल किया जाएगा। हम शोध प्रबंधन के संगठन का एक अवलोकन भी प्रस्तुत करते हैं और हमारे आवश्यक योगदान पर जोर देते हैं। अध्याय 2 में, हम सहज समय-निर्भर गुणांकों के साथ उत्कृष्टीय तरंग समीकरण से शुरू करते हैं और अच्छी तरह से प्रस्तुत परिणामों को विकसित करने के लिए आगे बढ़ते हैं। हालाँकि, हम अर्ध-उत्कृष्टीय मापक $\hbar \to 0$ की सीमा में अच्छी तरह से प्रस्तुत परिणाम भी पुनर्प्राप्त करते हैं। अध्याय 3 में, हमारे विश्लेषण को भिन्नात्मक लाप्लासियन को शामिल करने के लिए बढ़ाया गया है, और हम समष्टि-निर्भर बंधी हुई क्षमता के साथ भिन्नात्मक क्लेन-गॉर्डन समीकरण पर विचार करते हैं, और अच्छी तरह से स्थिति प्राप्त करते हैं, और पुनः प्राप्त करते हैं अर्ध-उत्कृष्टीय सीमा $\hbar \to 0$ में यूक्लिडियन समाधान। अध्याय 4 में, हम फिर अपने विश्लेषण को भिन्नात्मक तापीय समीकरण तक विस्तारित करते हैं, जिससे समय-निर्भर गुणांकों में अनियमितताएं होती हैं और प्रारंभिक कॉची आधार-सामग्री बहुपद रूप से बढ़ता है। हम असतत टेम्पर्ड वितरण के स्थान में नियमित गुणांक के मामले के लिए अच्छी तरह से प्रस्तुत परिणाम विकसित करते हैं और देखते हैं कि जब गुणांकों में वितरण विलक्षणता होती है तो बहुत कमजोर समाधान की धारणा ऐसे समीकरणों में कैसे अनुकूलित होती है। हम यह भी विश्लेषण करते

हैं कि अर्ध-उत्कृष्टीय सीमा $\hbar \rightarrow 0$ में बहुत कमजोर समाधानों का अनुमान कैसे लगाया जा सकता है। अध्याय 5 में, हम असतत गैर-नकारात्मक वर्ण-क्रम वाले संकारको के लिए तरंग समीकरण के एक विशेष वर्ग की जांच करते हैं, यानी, श्रो डिंगर संकारक से जुड़े तरंग समीकरण, और स्थिति को ध्यान में रखते हैं जब $|k| \rightarrow \infty$ के रूप में अनंत तक बढ़ जाता है तो विभव अनंत तक बढ़ता है। हम पहले असतत श्रो डिंगर संकारक के वर्ण-क्रम का अध्ययन करते हैं और साबित करते हैं कि जब विभव अनंत तक बढ़ता है तो इसमें विशुद्ध रूप से असतत वर्ण-क्रम होता है। हम विश्लेषण करते हैं कि इस मामले में अच्छी स्थिति के परिणाम उत्कृष्टीय तरंग समीकरण, भिन्नात्मक क्लेन-गॉर्डन समीकरण और भिन्नात्मक तापीय समीकरण के साथ कैसे भिन्न होते हैं। अध्याय 6 में, हम श्रो डिंगर संकारक के साथ सामान्य कैपुटो-प्रकार के समय-भिन्नात्मक प्रसार समीकरण के लिए अपनी जांच का विस्तार करते हैं और उस मामले पर विचार करते हैं जब $|k| \rightarrow \infty$ के रूप में अनंत तक बढ़ जाता है तो विभव अनंत तक बढ़ता है। शोध प्रबंधन के अंत में, हम कुछ शोध समस्याओं की ओर ध्यान आकर्षित करना चाहेंगे जो शोध प्रबंधन लिखने की प्रक्रिया के दौरान उभरी हैं।

Contents

List of Symbols

Symbol	**Meaning**
$\hbar$	Discretization parameter
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{N}_0$	Set of non-negative integers
$\mathbb{R}$	Set of real numbers
$\mathbb{T}$	Torus
$\mathbb{C}$	Set of complex numbers
S^n	Cartesian product of n-copies of set S
$\mathbb{N}_0^n$	$\mathbb{N}_0^n = \mathbb{N}^n \cup \{0\}$
$\|u\|_X$	Norm of u in normed linear space X
$A \Subset B$	A is a compact subset of B
$A \lesssim B$	If $\exists$ a constant $c > 0$ independent of appearing parameters such that $A \leq cB$
L^p	Lebesgue spaces, $1 \leq p \leq \infty$
H^s	Sobolev space of order s
C^α	Space of α-Hölder continuous function, $0 < \alpha < 1$
C^k	Space of k-times continuously differentiable functions
C^∞	Infinitely times differentiable functions
C_0^∞	Space of smooth functions with compact support

$\mathcal{D}'$	Space of distributions
γ^s	Gevrey space of order s