SOME STUDIES ON A GENERALIZED FOURIER EXPANSION FOR NONLINEAR AND NONSTATIONARY TIME SERIES ANALYSIS

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Dedicated to

MY TEACHERS

&

PARENTS

Smt. Ramlali Singh

Late Shri Shivadhar Singh

Certificate

This is to certify that the thesis entitled **"Some studies on a generalized Fourier expansion for nonlinear and nonstationary time series analysis"** being submitted by **Mr. Pushpendra Singh** to the **Department of Electrical Engineering, Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy** is the record of the bona-fide research work carried out by him under our supervision. In our opinion, the thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted either in part or in full to any other university or institute for the award of any degree or diploma.

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Abstract

The main objective of this thesis is to investigate and develop adaptive signal decomposition methods that yield a set of basis vectors, which are complete, orthogonal, local and adaptive. The instantaneous frequencies and square of amplitudes of these basis vectors produce a Time-Frequency-Energy (TFE) distribution, referred to as the Hilbert spectrum, that reveals the embedded structures of a signal. The main results of this thesis are briefly described in the following.

As is well known the Empirical Mode Decomposition (EMD) algorithm provides a general method for examining the TFE distribution of nonlinear and nonstationary time series. One of the important steps, in the EMD, is to obtain upper and lower envelopes by cubic spline interpolation of local extrema (maxima and minima) of a time series. Hence it is obvious that the performance of the algorithm would depend upon the type of interpolation and the locations of extrema in a time series. Deeper investigation leads to a novel nonpolynomial spline based EMD that reduces mode mixing and detrend uncertainty, which are some shortcomings of the EMD algorithm in the analysis of time series. A number of results on nonpolynomial interpolation, approximation and concomitant error analysis are presented. The generalized error analysis results for nonpolynomial, Hermite nonpolynomial interpolations and Taylor's nonpolynomial series approximation are derived. The EMD presented here improves the orthogonality among Intrinsic Mode Functions (IMFs) but does not make them completely orthogonal, therefore, energy is not preserved in the decomposition and there is always some energy leakage among IMFs.

We next propose algorithms which preserve energy in EMD. In the first Energy Preserving EMD (EPEMD) algorithm, a signal is decomposed into Linearly Independent (LI), Non Orthogonal yet Energy Preserving (LINOEP) IMFs and the residue. It is shown that

a vector in an inner product space can be represented as a sum of LI but non orthogonal vectors in such a way that Parseval's equality type property is satisfied. We first observe that from the set of 'n' IMFs, via the Gram-Schmidt Orthogonalization Method (GSOM), $n!$ sets of orthogonal functions can be obtained. In the second algorithm, we show that if the orthogonalization process proceeds from the lowest frequency IMF to the highest frequency IMF, then the GSOM yields functions which preserve the properties of IMFs and energy of the signal. However, all EMD algorithms are empirical in nature and do not provide a complete mathematical foundation required to fully understand the EMD. This motivates us to further explore and develop methods that capture all the relevant features and are also based on some sound mathematical foundation.

For many decades, there is a general perception in the literature that Fourier methods are not suitable for the analysis of nonlinear and nonstationary time series. We propose such a mathematical foundation in the form of a Fourier Decomposition Method (FDM) and demonstrate its efficacy for the analysis of nonlinear and nonstationary time series. The proposed FDM decomposes any data into a small number of 'Fourier Intrinsic Band Functions' (FIBFs). The FDM presents a generalized Fourier expansion with variable amplitudes and frequencies of a time series by the Fourier method itself. We also propose an idea of the zero-phase filter bank based Multivariate FDM (MFDM) algorithm for analysis of multivariate nonlinear and nonstationary time series. We also present an algorithm to obtain cutoff frequencies of zero-phase filter bank for MFDM.

Finally, we present a Fourier base method for the analysis and classification of electroencephalogram (EEG) signals using EEG rhythms. This method is based on MFMD method that analyzes common frequency components in multichannel EEG recordings. The Mean Frequency (MF) and RMS bandwidth are estimated from the EEG rhythms which are found to be statistically different, for the different classes of EEG recordings, and hence used to distinguish and classify the two classes of signals using a least-squares support vector machine (LS-SVM) classifier. Experimental results on a publicly available database demonstrate that the proposed feature sets effectively characterize the EEG rhythms and are suitable for classification between different classes of EEG signals.

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List of abbreviations and symbols

FS	Fourier Series
FT	Fourier Transform
STFT	Short-Time Fourier Transform
TF	Time–Frequency
IF	Instantaneous Frequency
FM	Frequency Modulation
AM	Amplitude Modulation
HT	Hilbert Transform
WVD	Wigner–Ville Distribution
CWT	Continuous Wavelet Transform
TFE	Time-Frequency-Energy
TFD	Time-Frequency Distribution
TFR	Time-Frequency Representation
FFT	Fast Fourier Transform
PCS	Polynomial Cubic Spline
NPCS	Nonpolynomial Cubic Spline
HNPCS	Hermite Nonpolynomial Cubic Spline
EMD	Empirical Mode Decomposition

EEMD	Ensemble EMD
MEMD	Multivariate EMD
MEMD	Orthogonal Multivariate EMD (OMEMD)
OEMD	Orthogonal EMD
EPEMD	Energy Preserving EMD
IMF	Intrinsic Mode Function
MIMF	Multivariate IMF
EPIMF	Energy Preserving IMF
FOIMF	Forward Orthogonal Intrinsic Mode Function
ROIMF	Revere Orthogonal Intrinsic Mode Function
ROUIMF	Revere Orthogonal and Uncorrelated Intrinsic Mode Function
DFT	Discrete Fourier Transform
IDFT	Inverse Discrete Fourier Transform
LINOEP	Linearly Independent Non Orthogonal yet Energy Preserving
GSOM	Gram-Schmidt Orthogonalization Method
FDM	Fourier Decomposition Method
FIBF	Fourier Intrinsic Band Function
MFIBF	Multivariate FIBF
MFDM	Multivariate FDM
AFIBF	Analytic FIBF
ZPFB	Zero-phase Filter Bank
FHS	Fourier-Hilbert Spectrum
MF	Mean Frequency

RMS	Root Mean Square.
RBW	Root Mean Square Bandwidth
EEG	Electroencephalography
IBF	Intrinsic Band Function
MIBF	Multivariate IBF
LS-SVM	Least-Squares Support Vector Machine
RBF	Radial Basis Function
ROC	Receiver Operating Characteristic
AUC	Area Under the ROC Curve
DSP	Digital Signal Processing
IIR	Infinite Impulse Response
FIR	Finite Impulse Response
C^∞ function	A C^∞ function (or smooth function) is a function that is differentiable for all degrees of differentiation.
ω	Based on the context, it is the frequency in *radians/seccond* and *radians/sample* for a continuous and discrete time signal, respectively.