On Time-L_1 Optimal Control of Linear Systems

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CERTIFICATE

This is to certify that the thesis entitled **"On Time-L_1 Optimal Control of Linear Systems"**, submitted by **Rajasree Sarkar** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** in Electrical Engineering, is a record of the original, bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree.

The results contained in this thesis have not been submitted either in part or in full to any other University or Institute for the award of any degree or diploma to the best of my knowledge.

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ABSTRACT

With the emergence of the concept of sparsity, the old classical problem of attaining time and fuel optimality has once again gained significant popularity in control system theory. Such problem referred to as the time-L_1 or time-fuel optimal control problem, has been widely explored in literature many decades ago. However, because of the inherent complexity of the problem, analytical solutions to the problem considering fuel optimality under various constraints of final time were derived only for simple class of linear time-invariant (LTI) systems like double integrators and other second order systems. Whereas, analytical characterization for general class of LTI systems is still under investigation. In this regard, this thesis considers the time-L_1 optimal control problem for state transfer of controllable single input linear time-invariant (LTI) systems with bounded inputs. Using the necessary conditions of Pontryagin's maximum principle (PMP), this thesis derives a combinatorial sequence characterization of the desired control for general LTI system such that only the knowledge of the system order is required to achieve possible sequences. The proposed characterization is then utilized to translate the original optimal problem into sets of equivalent static optimization problems or non-linear programs (NLPs). By performing such translation, a new method is devised to compute time-L_1 optimal control where discontinuities in the control are easily captured. For LTI systems with rational eigenvalues, this method is seen to provide global solution when solved using Lasarre's method based solvers like Gloptipoly, SparsePop etc. The thesis also presents an intermittent feedback scheme where system states are steered to within a user-defined safe region in finite time with reduced fuel consumption. Such intermittent schemes are useful in scenarios where system is subjected to external disturbances or when analytical feedback solution is difficult to achieve. In addition, the thesis presents a time-L_1 efficient approach in the context of multi-agent system and derives a decentralized scheme for consensus tracking in finite time using reduced amount of fuel.

Keywords: Time-L_1 Optimal Control, Bang-off-bang Control, Combinatorial Sequence Characterization, Intermittent Feedback Control, Time-L_1 Optimal Pursuit-Evasion Policy, Decentralized Consensus Tracking, Sparsity, Resource-constrained System.

सार

विरलता की अवधारणा के उद्भव के साथ, समय और ईंधन इष्टतमता प्राप्त करने की पुरानी शास्त्रीय समस्या ने एक बार फिर नियंत्रण प्रणाली सिद्धांत में महत्वपूर्ण लोकप्रियता प्राप्त की है। इस तरह की समस्या को समय-एल$_1$ या समय-ईंधन इष्टतम नियंत्रण समस्या के रूप में जाना जाता है, कई दशकों पहले साहित्य में व्यापक रूप से खोजा गया है। हालाँकि, समस्या की अंतर्निहित जटिलता के कारण, अंतिम समय की विभिन्न बाधाओं के तहत ईंधन इष्टतमता पर विचार करने वाली समस्या का विश्लेषणात्मक समाधान केवल रैखिक समय-अपरिवर्तनीय (एलटीआई) प्रणालियों के सरल वर्ग जैसे डबल इंटीग्रेटर्स और अन्य द्वितीय क्रम प्रणालियों के लिए प्राप्त किया गया था। जबकि, एलटीआई सिस्टम के सामान्य वर्ग के लिए विश्लेषणात्मक लक्षण वर्णन अभी भी जांच के दायरे में है। इस संबंध में, यह थीसिस सीमित इनपुट के साथ नियंत्रणीय एकल इनपुट लीनियर टाइम-इनवेरिएंट (एलटीआई) सिस्टम के राज्य हस्तांतरण के लिए समय-एल$_1$ इष्टतम नियंत्रण समस्या पर विचार करता है। पोंट्रीगिन के अधिकतम सिद्धांत (पीएमपी) की आवश्यक शर्तों का उपयोग करते हुए, यह थीसिस सामान्य एलटीआई प्रणाली के लिए वांछित नियंत्रण के संयोजन अनुक्रम लक्षण वर्णन को प्राप्त करती है, जैसे कि संभावित अनुक्रमों को प्राप्त करने के लिए केवल सिस्टम ऑर्डर का ज्ञान आवश्यक है। प्रस्तावित लक्षण वर्णन का उपयोग मूल इष्टतम समस्या को समतुल्य स्थैतिक अनुकूलन समस्याओं या गैर-रैखिक कार्यक्रमों (एनएलपी) के सेट में अनुवाद करने के लिए किया जाता है। इस तरह के अनुवाद को करने से, समय की गणना करने के लिए एक नई विधि तैयार की जाती है-एल$_1$ इष्टतम नियंत्रण जहां नियंत्रण में असंततता आसानी से पकड़ी जाती है। तर्कसंगत इंजन वैल्यूज के साथ एलटीई सिस्टम के लिए, इस पद्धति को वैश्विक समाधान प्रदान करने के लिए देखा जाता है, जब लैसेरर की विधि आधारित सॉल्वर जैसे ग्लोप्तीपॉली, स्पार्सपॉप आदि का उपयोग करके हल किया जाता है। थीसिस एक आंतरायिक प्रतिक्रिया योजना भी प्रस्तुत करती है, जहाँ सिस्टम स्टेट्स को एक उपयोगकर्ता-परिभाषित सुरक्षित क्षेत्र के भीतर चलाया जाता है। कम ईंधन की खपत के साथ सीमित समय। ऐसी आंतरायिक योजनाएँ उन परिदृश्यों में उपयोगी होती हैं जहाँ सिस्टम बाहरी गड़बड़ी के अधीन होता है या जब विश्लेषणात्मक प्रतिक्रिया समाधान प्राप्त करना कठिन होता है। इसके अलावा, थीसिस मल्टी-एजेंट सिस्टम के संदर्भ में एक समय-एल$_1$ कुशल दृष्टिकोण प्रस्तुत करती है और ईंधन की कम मात्रा का उपयोग करके सीमित समय में आम सहमति पर नज़र रखने के लिए एक विकेन्द्रीकृत योजना प्राप्त करती है।

कीवर्ड: समय-एल$_1$ इष्टतम नियंत्रण, बैंग-ऑफ-बैंग नियंत्रण, संयोजन अनुक्रम विशेषता, आंतरायिक प्रतिक्रिया नियंत्रण, समय-एल$_1$ इष्टतम पीछा-अपवंचन नीति, विकेंद्रीकृत आम सहमति ट्रैकिंग, विरलता, संसाधन-विवश प्रणाली।

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Nomenclature

Acronyms	**Description**
CPT	Control Parameterization Technique
DCF	Diagonal Canonical Form
DST	Directed Spanning Tree
GMP	Generalized Moment Problem
IPOPT	Interior Point Optimizer
JCF	Jordon Canonical Form
LQR	Linear Quadratic Regulator
LTI	Linear Time-invariant
MAS	Multi-agent System
MINLP	Mixed-integer Non-linear Programs
MPC	Model Predictive Control
NLP	Non-linear Program
OP	Optimization Problem (Static)
P-E	Pursuer-evader
PMP	Pontraygin's Minimum Principle
PO	Polynomial Optimization Problem
SDP	Semi-definite Program
SNOPT	Sparse Non-linear Optimizer

Continued from previous page

Acronyms	**Description**
TFCA	Time-fuel Candidates Automata

List of Symbols	**Description**
$\|\|x(t)\|\|_p$	L_p norm of vector $x(t)$ for $p \in [0, \infty]$
$\|x(t)\|_1$	L_1 norm of vector $x(t)$
$\|\|\mathbf{A}\|\|_2$	Induced L_2 norm of matrix $\mathbf{A}$ i.e. $\|\mathbf{A}\| = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|}$
$\mathbb{R}$	Set of Real numbers
$\mathbb{Q}$	Set of Rational numbers
$\mathbb{C}$	Set of Complex numbers
$\mathbb{Z}$	Set of Integers
sgn	Signum Function
$\lambda(\mathbf{A})$	Eigenvalues of $\mathbf{A}$
diag()	Diagonal matrix
H	Hamiltonian
$\mathcal{N}(S)$	Cardinality of set S
$u^*(t)$	Optimal input
$\bar{S}$	Conjugate set of S
$\langle \cdot \rangle$	Inner Product
$\{a, b, c\}$	Sequence a $\rightarrow$ b $\rightarrow$ c
$\cup$	Union
$\subseteq$	Subset
$\nsubseteq$	Not subset

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List of Symbols	**Description**
$\emptyset$	Null set
${}^nC_r$	Combination
conv()	Convex hull of a set
h-representation	Half-space representation
g-representation	Generator representation
a.e.	Almost Everywhere
$\mathcal{A}_i$	Attainable set to i
$\mathcal{R}_i$	Reachable set to i
T_s	Sampling time
inf	Infimum
sup	Supremum
$\max(a, b)$	Maximum between a and b
$\min(a, b)$	Minimum between a and b
argmin	Minimum argument