

On Derivations of Rings and Group Rings

By

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Dedicated
To
My Family

Certificate

It is certify that the thesis entitled "*On Derivations of Rings and Group Rings*" presented by Mr. Balchand Prajapati (2005MAZ8113) is worthy of consideration for the award of the degree of Doctor of Philosophy and is a record of the original bonafide research work carried out by him under my guidance and supervision and the results contained in it have not been submitted in part or full to any other University or Institute for award of any degree/diploma.

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Abstract

In the theory of derivations, additive mappings play a vital role. In this thesis, we study n -additive mappings on prime rings and show that if it satisfies some identity then it has some particular form. In this thesis, we also study derivations on prime rings, semiprime rings and group rings. A ring R is said to be prime if for $x, y \in R$, $xRy = 0$ implies that $x = 0$ or $y = 0$ and R is said to be semiprime if for $x \in R$, $xRx = 0$ implies that $x = 0$. It is clear that every prime ring is semiprime but not conversely. An additive mapping $d : R \rightarrow R$ is said to be a derivation (Jordan derivation) if $d(xy) = xdy + (dx)y$ (resp. $dx^2 = xdx + (dx)x$) for all $x, y \in R$. Every derivation is a Jordan derivation but the converse need not be true. An additive mapping $g_d : R \rightarrow R$ is said to be a generalized derivation (generalized Jordan derivation) associated with a (Jordan) derivation $d : R \rightarrow R$ if $g_d(xy) = (g_dx)y + xdy$ (resp. $g_dx^2 = (g_dx)x + xdx$) for all $x, y \in R$. Every generalized derivation is a generalized Jordan derivation but again the converse need not be true. Several algebraists, for example, Herstein, Ashraf, Rehman, Cusack etc. worked in this direction and proved that if a ring is prime or semiprime of characteristic not equal to 2 or any ring of characteristic not equal to 2 and having a commutator which is not a zero

divisor then the converse also holds true. An additive mapping $g_d : R \rightarrow R$ is said to be a generalized left derivation (generalized Jordan left derivation) associated with a (Jordan) left derivation $d : R \rightarrow R$ if $g_d(xy) = xg_dy + ydx$ (resp. $g_dx^2 = xg_dx + xdx$) for all $x, y \in R$. Here also it is clear that every generalized left derivation is also a generalized Jordan left derivation but the converse need not be true. Ashraf *et. al.*, [2], proved that if a ring is prime of characteristic not equal to 2 then the converse is also true. In this thesis, we prove that the converse holds true if the ring is semiprime of characteristic not equal to 2 in which $x^2 = 0$ implies $x = 0$.

The collection of all derivations does not form a ring but it forms a Lie ring. Jordan *et. al.*, [40, 42], showed that if a ring is prime (semiprime) of characteristic not equal to 2 then the Lie ring of derivations is a prime (semiprime) Lie ring. We prove that the collection of all generalized derivations, for a prime (semiprime) ring of characteristic not equal to 2, forms a prime (semiprime) Lie ring. We also prove that the Lie ring of generalized inner derivations, on the contrary of Lie ring of inner derivations, is not a prime Lie ring.

The derivation on group rings was first initiated by Smith, [53]. After this important contributions are made by Ferrero *et. al.*, [27] and Spiegel, [54]. In this thesis, we prove that every derivation d on a group ring RG , for R , a semiprime ring and G , an abelian group such that $dG = 0$, can be uniquely extended to a derivation on Q_lG , where Q_l is the left Martindale's ring of quotient of R . We prove that every higher derivation on a group algebra FG , F , a field and G , a group such that $[G : \rho(G)] < \infty$, $\rho(G)$, the finite conjugate subgroup of G , can be extended to a higher derivation on the classical ring of quotient Q_C of FG .

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