

GRAPH COLORING AND ITS VARIATIONS: STRUCTURAL AND ALGORITHMIC STUDY

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**GRAPH COLORING AND ITS VARIATIONS:
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by

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Dedicated to
My Family

Certificate

This is to certify that the thesis entitled **“Graph Coloring and Its Variations: Structural and Algorithmic study”** submitted by **“Ms. Shaily Verma”** to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

Graph coloring is one of the important and widely studied areas of research in graph theory. A proper k -coloring of a graph G is an assignment of k colors to the vertices of G such that no two adjacent vertices receive the same color. In this thesis, we study the algorithmic aspects of four variations of graph coloring, namely (i) Grundy coloring, (ii) partial Grundy coloring, (iii) total coloring, and (iv) adjacent vertex distinguish (AVD)-total coloring.

A Grundy k -coloring of a graph G is a proper k -coloring such that for every vertex v of color i , v has a neighbor of color j , for all $j < i$. Any Grundy coloring of a graph G is a proper coloring generated by the well known heuristic, *greedy coloring algorithm*, which takes an ordering of the vertices, $(v_1, v_2, \dots, v_n)$ and assign v_i the least color which is not assigned to any neighbor v_j of v_i , for $j < i$. The Grundy number of a graph G is the maximum number of colors used by the greedy coloring algorithm to properly color G . A partial Grundy coloring is a proper coloring such that for every color i , there exists a vertex of color i having at least one neighbor of color j , for all $j < i$ and the maximum number of colors used for any partial Grundy coloring of a graph G is called the partial Grundy number of G .

In this thesis, we strengthen the existing NP-completeness results for the problems to find the Grundy number and the partial Grundy number for bipartite graphs

by giving the NP-completeness results for these problems in some subclasses of bipartite graphs. On the other hand, we propose some polynomial time algorithms to solve these problems in a few subclasses of bipartite graphs. In this thesis, we improve an existing upper bound on the partial Grundy number. We also give an NP-completeness result for the problem to find the partial Grundy number, for doubly chordal graphs and give a linear time algorithm for split graphs. Further, we prove that the problem remains NP-complete for the complement of bipartite graphs.

In the other part of the thesis, we study total coloring and AVD-total coloring in various graph classes. A total coloring is an assignment of colors to the vertices and edges of the graph such that no two adjacent vertices or edges receive the same color and the end vertices of an edge receive different colors. An AVD-total coloring is a total coloring of a given graph G such that for any adjacent vertices u and v , the color set of u and v are different, where the color set of a vertex contains the colors of the edges incident on that vertex and the color of the vertex itself. The minimum number of colors used for a total (AVD-total) coloring of a given graph G is called the total (AVD-total) chromatic number of G . The total chromatic number of a graph G is at least $\Delta(G) + 1$, and it is conjectured that it is at most $\Delta(G) + 2$, which is known as *total coloring conjecture*.

It is known that the problem to classify the graphs according to their total chromatic number is NP-hard even for 3-regular bipartite graphs. In this thesis, we study the total chromatic number in two subclasses of bipartite graphs, namely, biconvex graphs and chain graphs. In this work, we also consider the graph operations: central graphs and corona products. We prove that the total coloring conjecture holds for the central graph of a graph. We study the AVD-total chromatic number of the central graphs of some basic graph classes and obtain some results on the AVD-total chromatic number of corona products of graphs. We also give some results on AVD-total coloring of split graphs.

सार

ग्राफ़ कलरिंग (Graph Coloring) ग्राफ़ सिद्धांत (Graph Theory) में अनुसंधान के महत्वपूर्ण और व्यापक रूप से अध्ययन किए गए क्षेत्रों में से एक है। एक ग्राफ़ (Graph) G की एक उचित (proper) k -कलरिंग शीर्षों (vertices) को रंग सौंपने का कार्य है, जिसमें कोई भी दो आसन्न (adjacent) शीर्ष समान रंग प्राप्त नहीं करते हैं। इस निबंध (thesis) में, हम ग्राफ़ कलरिंग के चार रूपों के कलन विधि (algorithmic) पहलुओं, (i) ग्रन्डी कलरिंग (Grundy Coloring), (ii) आंशिक ग्रन्डी कलरिंग (partial Grundy coloring), (iii) कुल कलरिंग (total coloring), और (iv) निकटवर्ती शीर्ष भेद (एवीडी)-कुल कलरिंग (adjacent vertex distinguish (AVD)-total coloring) का अध्ययन करते हैं।

एक ग्राफ़ G का ग्रन्डी k -कलरिंग एक उचित k -कलरिंग है जिसमें $j < i$ के लिए, हर शीर्ष v जिसका रंग i है, तो उस v के पास रंग j का पड़ोसी है। ग्राफ़ G की ग्रन्डी कलरिंग एक उचित कलरिंग है जिसे प्रसिद्ध अनुमानी, लालची कलरिंग कलन विधि (heuristic, greedy coloring algorithm) द्वारा उत्पन्न किया गया है, जो $j < i$ के लिए, शीर्षों का एक क्रम लेता है, $(v_1, v_2, \dots, v_n)$ और v_i को सबसे कम रंग प्रदान करता है जो किसी भी पड़ोसी v_j को नहीं सौंपा गया है। एक ग्राफ़ G की ग्रन्डी संख्या (Grundy number), लालची रंग कलन विधि द्वारा ग्राफ़ की उचित कलरिंग करने के लिए रंगों की अधिकतम संख्या है। आंशिक ग्रन्डी कलरिंग (Partial Grundy coloring) एक उचित कलरिंग है जिसमें अगर $j < i$ है तो, हर रंग i के लिए, उस रंग i का एक शीर्ष मौजूद होता है, जिसके पास रंग j का कम से कम एक पड़ोसी है। ग्राफ़ G के किसी भी आंशिक ग्रन्डी कलरिंग के लिए उपयोग किए जाने वाले रंगों की अधिकतम संख्या को G की आंशिक ग्रन्डी संख्या (partial Grundy number) कहा जाता है।

इस निबंध में, हम बाइपार्टाइट ग्राफ़ (bipartite graphs) के कुछ उपवर्गों में ग्रन्डी संख्या और आंशिक ग्रन्डी संख्या खोजने की समस्या के लिए NP-पूर्णता (NP-completeness) के परिणामों को प्राप्त कर, मौजूदा NP-पूर्णता के परिणामों को मजबूत करते हैं। दूसरी ओर, हम कुछ बहुपद समय कलन विधि (polynomial time algorithm) का प्रस्ताव करते हैं ताकि इन समस्याओं को बाइपार्टाइट ग्राफ़ के कुछ उपवर्गों के लिए हल किया जा सके। इस निबंध में, हम आंशिक ग्रन्डी संख्या पर मौजूदा ऊपरी सीमा (upper bound) में सुधार करते हैं। दोहरे कोरल ग्राफ़ (doubly chordal graphs) के लिए आंशिक

ग्रन्डी संख्या खोजने की समस्या के लिए, हम NP-पूर्णता का एक परिणाम भी देते हैं और स्प्लिट ग्राफ (split graphs) के लिए एक-रेखीय समय कलन विधि (linear time algorithm) प्रस्तुत करते हैं। इसके अलावा, हम साबित करते हैं कि बाइपार्टाइट ग्राफ के पूरक (complement) के लिए समस्या NP-पूर्ण ही बनी रहती है।

निबंध के दूसरे भाग में, हम विभिन्न ग्राफ वर्गों के लिए कुल कलरिंग और AVD-कुल कलरिंग का अध्ययन करते हैं। कुल कलरिंग शीर्षों और ग्राफ के किनारों (edges) को रंग सौंपने का एक कार्य है ताकि कोई दो आसन्न शीर्षों या किनारों को एक ही रंग प्राप्त नहीं हो और किसी किनारे के अंतिम शीर्ष को अलग रंग प्राप्त हो। किसी दिए गए ग्राफ G के लिए AVD-कुल कलरिंग एक ऐसी कुल कलरिंग है जिसमें किसी भी आसन्न शीर्षों u और v के लिए, u और v के रंग सेट (color set) अलग-अलग होते हैं, जहाँ एक शीर्ष के रंग सेट में उस शीर्ष पर पड़े (incident) किनारों के रंग होते हैं और साथ में स्वयं शीर्ष का रंग होता है। किसी दिए गए ग्राफ G के कुल (AVD-कुल) कलरिंग के लिए उपयोग किए जाने वाले रंगों की न्यूनतम संख्या को G का कुल (AVD-कुल) क्रोमेटिक संख्या (total (AVD-total) chromatic number) कहा जाता है। G की कुल क्रोमेटिक संख्या कम से कम $\Delta(G) + 1$ होती है, और यह अनुमान लगाया जाता है कि कुल क्रोमेटिक संख्या ज्यादा से ज्यादा $\Delta(G) + 2$ है, जिसे कुल कलरिंग कन्जेक्चर (total coloring conjecture) के नाम से जाना जाता है।

यह ज्ञात है कि 3-नियमित बाइपार्टाइट ग्राफ (3-regular bipartite graphs) को उनके कुल क्रोमेटिक संख्या के अनुसार वर्गीकृत करने की समस्या भी NP-हार्ड है। इस निबंध में, हम बाइपार्टाइट ग्राफ के दो उपवर्गों: बाइकॉन्वेक्स ग्राफ (biconvex graphs) और चैन ग्राफ (chain graphs) के कुल क्रोमेटिक संख्या का अध्ययन करते हैं। इस काम में, हम निम्न ग्राफ संक्रियाओं (graph operations) पर भी विचार करते हैं: केंद्रीय ग्राफ (central graphs) और कोरोना उत्पाद (corona products)। हम साबित करते हैं कि कुल कलरिंग कन्जेक्चर एक ग्राफ के केंद्रीय ग्राफ के लिए सत्य है। हम कुछ बुनियादी ग्राफ वर्गों के केंद्रीय ग्राफ के AVD-कुल क्रोमेटिक संख्या का अध्ययन करते हैं और ग्राफ के कोरोना उत्पादों के AVD-कुल क्रोमेटिक संख्या पर कुछ परिणाम प्राप्त करते हैं। हम स्प्लिट ग्राफ के AVD-कुल कलरिंग पर कुछ परिणाम भी देते हैं।

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List of Symbols

Symbol	Meaning
$\mathbb{N}$	Set of <i>natural numbers</i>
$\forall x$	For <i>all</i> x
$x \in X$	x is a <i>member</i> of X
$A \subseteq X$	A is a <i>subset</i> of X
$ X $	The <i>cardinality</i> of a set X
$A \cap B$	The <i>intersection</i> of A and B
$A \cup B$	The <i>union</i> of A and B
$A \setminus B$	The <i>set difference</i> of A and B
$A \times B$	The <i>Cartesian product</i> of A and B
$\emptyset$	The <i>empty</i> set
$\square$	The <i>end of a proof</i>
$ $	such that
$E(G)$	The set of <i>edges</i> of G
$V(G)$	The set of <i>vertices</i> of G
$d(v)$	The <i>degree</i> of the vertex v in G
Δ or $\Delta(G)$	The <i>maximum degree</i> in G
δ or $\delta(G)$	The <i>minimum degree</i> in G

$N(v)$	<i>neighborhood</i> of v in G
$N[v]$	<i>closed neighborhood</i> of v in G
$G[H]$	The subgraph of G <i>induced</i> by the vertices of H
$\bar{G}$	The <i>complement</i> of an undirected graph G
$G_1 \circ G_2$	The <i>corona product</i> of the graphs G_1 and G_2
$G_1 \cong G_2$	Graphs G and G' are isomorphic
$\omega(G)$	The <i>clique number</i> of G
K_n	The <i>complete graph</i> on n vertices
C_n	The <i>cycle</i> on n vertices
P_n	The <i>path</i> on n vertices
$K_{m,n}$	The <i>complete bipartite graph</i> on $m + n$ vertices, which can be partitioned into an m -independent set and an n -independent set
$\chi(G)$	The <i>chromatic number</i> of G
$\chi'(G)$	The <i>chromatic index</i> of G
$\Gamma(G)$	The <i>Grundy number</i> of G
$\partial\Gamma(G)$	The <i>partial Grundy number</i> of G
$\zeta(G)$	The <i>stair factor</i> of G
$\chi''(G)$	The <i>total chromatic number</i> of G
$\chi''_a(G)$	The <i>AVD-total chromatic number</i> of G
$\lambda'(G)$	The <i>edge domination number</i> of G
P	The class of <i>deterministic</i> polynomial-time solvable problems
NP	The class of <i>nondeterministic</i> polynomial-time solvable problems