A THESIS ON

OPTIMAL RULES OF NUMERICAL INTEGRATION FOR ANALYTIC FUNCTIONS

By

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CERTIFICATE

This is to certify that the thesis entitled "Optimal Rules of Numerical Integration for Analytic Functions" which is being submitted by Mr. Bushan Lal Raina for the award of Doctor of Philosophy (Mathematics) to the Indian Institute of Technology, Delhi, is a record of bonafide research work. He has worked for the last three years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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SYNOPSIS

Classical rules for the numerical integration of a function were developed so as to be interpolatory for certain convenient sets of functions, mainly the powers of the independent variable. Recently, attempts have been made to obtain the so-called 'optimal' rules, determining abscissas and weights by minimizing the norm of the error of the rule over a Hilbert space. Various such rules have been computed numerically over Hilbert spaces of functions analytic in circles and certain ellipses. In the present work an attempt has been made to obtain these optimal rules explicitly over suitable Hilbert spaces of analytic functions and to study the properties of these rules. The explicit determination of an optimal rule, as compared to its numerical computation, makes it possible to study the advantages of these rules over the classical rules. The main idea has been to employ a Hilbert space possessing a complete orthonormal sequence the elements of which are also orthogonal on the range of integration with respect to the weight function in the integral itself. Such Hilbert spaces of analytic functions are available only for the four Jacobi weight functions $(1-x)^{\pm\frac{1}{2}}(1+x)^{\pm\frac{1}{2}}$ and, therefore, our study of optimal rules for numerical integration is concerned only with integrals with these four weight functions.

The thesis consists of six chapters and a brief description of each chapter follows.

In Chapter I we survey the work done so far for the construction of optimal rules. We then discuss briefly the results obtained in the thesis.

In Chapter II we discuss 'intermediate-optimal' quadratures for the weighted integral $L(f) = \int_{-1}^{1} w(x)f(x)dx$ with $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $(\alpha,\beta) = (\pm \frac{1}{2}, \pm \frac{1}{2})$ and with f analytic on $[-1,1]$. The abscissas are preassigned at the zeros of the orthogonal polynomial corresponding to $w(x)$ and the intermediate-optimal weights are determined so as to minimize the norm of the error over a 'suitable' Hilbert space $H^2(\mathcal{E}_\rho ;|w(z)|)$ of functions analytic in an ellipse $\mathcal{E}_\rho$ with foci at $(1,0)$ and $(-1,0)$ and whose sum of the semiaxes is ρ. The Hilbert space $H^2(\mathcal{E}_\rho;|w(z)|)$ results from a weighted line integral around $\mathcal{E}_\rho$ with the same weight as that in the integral itself. It is only for the above four Jacobi weight functions $w(x)$ that a system of polynomials doubly orthogonal, with respect to $w(x)$ on $[-1,1]$ and with respect to $|w(x)|$ on the line integral around $\mathcal{E}_\rho$, is known. The intermediate-optimal weights are given explicitly and these come out proportional to the corresponding Gaussian weights. An intermediate-optimal rule is thus simply related to the corresponding Gaussian rule; the error norms of intermediate-optimal and the corresponding Gaussian rules can be easily compared. The weights in an n-point intermediate-optimal rule differ from the corresponding Gaussian weights by at least $O(\rho^{-4n})$ for $\rho \to \infty$.

In Chapter III we consider optimal rules for $L(f)$ over $H^2(\mathcal{E}_\rho; |w(z)|)$; the optimal abscissas and weights are determined so as to minimize the norm of the error over $H^2(\mathcal{E}_\rho; |w(z)|)$. For $w(x) = (1-x^2)^{-\frac{1}{2}}$ the optimal abscissas and weights have been determined explicitly; the optimal abscissas are the same as the Gaussian abscissas and the optimal weights are proportional to the corresponding Gaussian weights. For the other three $w(x)$, an explicit determination of the optimal abscissas and weights did not seem possible; and so, we have studied the 'closeness' of intermediate-optimal and the corresponding optimal rules.

In Chapter IV we discuss optimal rules for $L(f)$ over $H^2(\mathcal{E}_\rho; |w(z)|)$ with one or two of the abscissas fixed at -1 or $+1$. For $w(x) = (1-x^2)^{-\frac{1}{2}}$ we obtain explicitly the three optimal quadratures of the closed type with one abscissa fixed at $+1$ or -1 and two abscissas fixed at ± 1. The optimal abscissas are the same as in the corresponding Gaussian quadrature while the optimal weights are proportional to the corresponding Gaussian weights. The error norms of these optimal rules can again be compared with the error norms of the corresponding Gaussian rules.

In Chapter V we discuss properties of optimal and intermediate-optimal rules for L(f) over $H^2(\mathcal{E}_\rho;|w(z)|)$. We also consider the construction of optimal and intermediate-optimal rules for L(f) under the side condition that the rules be exact for constants. The optimal rules without precision for constants are simply related to the corresponding optimal rules with precision for constants; this makes the construction of the latter rules simpler once the former rules have been computed. For $w(x) = (1-x^2)^{-\frac{1}{2}}$, Optimal rule with precision for constants is identical with the corresponding Gaussian rule. For the remaining three w(x), an intermediate-optimal quadrature with precision for constants is identical with the corresponding Gaussian quadrature. For these three w(x) we have studied the 'closeness' of the optimal and intermediate-optimal quadratures with precision for constants.

In Chapter VI we extend the above results to discuss optimal and intermediate-optimal rules for the double integral $\int_{-1}^{1}\int_{-1}^{1} u(x)\, v(y)\, f(x,y)dx\, dy$ with u(x) , v(x) $= (1-x)^\alpha(1+x)^\beta$, $(\alpha,\beta) = (\pm \frac{1}{2}, \pm \frac{1}{2})$ and with f analytic in $\{(x,y): -1 \leqslant x \leqslant 1, \ -1 \leqslant y \leqslant 1 \}$. The Hilbert space employed is $H^2(\mathcal{E}_\rho \times \mathcal{E}_\rho; |u(z)v(w)|)$ resulting from double line integral around $\mathcal{E}_\rho \times \mathcal{E}_\rho$ and with the same weight as in the given integral. The results are parallel to those obtained for quadratures in Chapters II - V.

Some of the results of Chapters II, III and IV have been published in Chawla and Raina [16].

CONTENTS

Note: A reference to Th. n of Chapter m is made as Th. m.n, Same for Lemmas, etc. Similarly, Eqn. (m.n) means Eqn. numbered n of Chapter m.