POINTWISE A POSTERIORI ERROR ANALYSIS OF FINITE ELEMENT METHODS FOR ELLIPTIC VARIATIONAL INEQUALITIES

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by

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in fulfillment of the requirements of the degree of Doctor of Philosophy
to the

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Dedicated to
my grandmother

Certificate

This is to certify that the thesis entitled **POINTWISE A POSTERIORI ERROR ANALYSIS OF FINITE ELEMENT METHODS FOR ELLIPTIC VARIATIONAL INEQUALITIES** submitted by **Mr. Rohit Khandelwal** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
November 2022

Prof. Kamana Porwal
Indian Institute of Technology Delhi

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This thesis marks the endings and beginnings of many things in my life. The conclusion results from many hours of hard work, feeling lost and finding myself again. This Ph.D. has helped instil my faith in the fact if you approach a problem from multiple angles, the solution is bound to appear sooner or later.

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Rohit Khandelwal

Abstract

This thesis primarily deals with the pointwise a posteriori error analysis of finite element methods for the elliptic variational inequalities. Therein, we focus on two contact problems- the Signorini problem and the Obstacle problem- which are the prototype models for the elliptic variational inequality of the first kind. The adaptive finite element methods generally are essential for contact problems as the solution of these problems exhibits singular behaviour around the free boundary due to loss of regularity. The error estimates in the supremum norm are important for variational inequalities since they provide localized information on the approximation. The supremum norm a posteriori estimates enable locating the singularities locally to control the pointwise errors. Pointwise error control is rather challenging to carry out due to the variational nature of the finite element method. Moreover, for instance, the pointwise error control appears to be decisive for certain variational inequalities where the computed solution serves as a physical quantity as it determines the pointwise accuracy.

This thesis consists of seven chapters including the introductory chapter and the conclusion. In Chapter 1, we review the literature and collect some known results and preliminaries, which play a crucial role in the next chapters. In Chapter 2, we derive the pointwise a posteriori error estimates of the Signorini problem using linear conforming finite element methods. The analysis mainly depends on the super and sub solutions corresponding to the continuous solution $\boldsymbol{u}$ and standard a priori estimates for the Green's matrix of the divergence type operator. In Chapter 3, a posteriori error estimate for a class of discontinuous Galerkin (DG) methods has been derived in the supremum norm for the Signorini problem. An enriching map from the DG finite element space to conforming finite element space has been constructed and used correctly in the analysis of Chapter 3. We develop a new residual-based pointwise a posteriori error estimator for the unilateral contact problem using the quadratic conforming finite element method (FEM) in Chapter 4. In the analysis, we work with the continuous piecewise quadratic finite element space and continuous piecewise linear multipliers on the contact zone. In Chapter 5, we derive the pointwise a posteriori error analysis of the quadratic finite element method for the elliptic obstacle problem. The reliability and the efficiency of the proposed a posteriori error estimator are discussed. The bounds on the regularized Green's function and a sign of the discrete Lagrange multiplier play a crucial role in the analysis. In Chapter 6, we perform a posteriori error analysis in the supremum norm for the quadratic DG methods for the obstacle problem. We define two discrete sets (motivated by Gaddam, Gudi and Kamana [44]), one with integral constraints and another with the nodal constraints at the quadrature points, and discuss the pointwise reliability and efficiency of the proposed a posteriori error estimator. In the analysis, we employ a linear averaging function to transfer DG finite element space to standard conforming finite element space and exploit the sharp bounds on the Green's function of the Poisson problem. Several numerical experiments are presented in Chapter 2 to Chapter 6, numerically illustrating the theoretical order of convergence derived in the analysis. In Chapter 7, we summarize the work done and the possible extensions with scope for future investigations are proposed.

सार

यह शोध प्रबंध मुख्य रूप से एल्लिप्टिक परिवर्तनशील असमानताओं के लिए परिमित तत्व विधियों के बिंदुवार एक पश्चवर्ती त्रुटि विश्लेषण से संबंधित है। इसमें, हम दो संपर्क समस्याओं पर ध्यान केंद्रित करते हैं– सिग्नोरिनी समस्या और बाधा समस्या– जो पहली तरह की दीर्घवृत्तीय परिवर्तनशील असमानता के लिए प्रोटोटाइप मॉडल हैं। संपर्क समस्याओं के लिए अनुकूली परिमित तत्व विधियां आम तौर पर आवश्यक होती हैं क्योंकि इन समस्याओं का समाधान नियमितता के नुकसान के कारण मुक्त सीमा के आसपास एकवचन व्यवहार प्रदर्शित करता है। सर्वोच्च मानदंड में त्रुटि अनुमान परिवर्तनशील असमानताओं के लिए महत्वपूर्ण हैं क्योंकि वे सन्निकटन पर स्थानीयकृत जानकारी प्रदान करते हैं। सुप्रीमम मानदंड एक पश्चगामी अनुमान बिंदुवार त्रुटियों को नियंत्रित करने के लिए स्थानीय रूप से विलक्षणताओं का पता लगाने में सक्षम बनाता है। परिमित तत्व विधि की परिवर्तनशील प्रकृति के कारण बिंदुवार त्रुटि नियंत्रण बल्कि चुनौतीपूर्ण है। इसके अलावा, उदाहरण के लिए, बिंदुवार त्रुटि नियंत्रण कुछ भिन्नता असमानताओं के लिए निर्णायक प्रतीत होता है जहां गणना समाधान भौतिक मात्रा के रूप में कार्य करता है क्योंकि यह बिंदुवार सटीकता निर्धारित करता है।

इस थीसिस में परिचयात्मक अध्याय और निष्कर्ष सहित सात अध्याय हैं। अध्याय 1 में, हम साहित्य की समीक्षा करते हैं और कुछ ज्ञात परिणाम और प्रारंभिक तैयारी एकत्र करते हैं, जो अगले अध्यायों में महत्वपूर्ण भूमिका निभाते हैं। अध्याय 2 में, हम रेखीय अनुरूप परिमित तत्व विधियों का उपयोग करके सिग्नोरिनी समस्या के बिंदुवार एक पश्चवर्ती त्रुटि अनुमान प्राप्त करते हैं। विश्लेषण मुख्य रूप से निरंतर समाधान u के अनुरूप सुपर और उप समाधानों पर निर्भर करता है और विचलन प्रकार ऑपरेटर के ग्रीन के मैट्रिक्स के लिए एक प्राथमिक अनुमान मानक है। अध्याय 3 में, असतत गैलेरकिन (डीजी) विधियों के एक वर्ग के लिए एक पश्चवर्ती त्रुटि अनुमान सिग्नोरिनी समस्या के लिए सर्वोच्च मानदंड में प्राप्त किया गया है। डीजी परिमित तत्व स्थान से अनुरूप परिमित तत्व स्थान तक एक समृद्ध मानचित्र का निर्माण किया गया है और अध्याय 3 के विश्लेषण में सही ढंग से उपयोग किया गया है। हम अध्याय 4 में द्विघात अनुरूप परिमित तत्व विधि (एफ ई एम) का उपयोग करके एकतरफा संपर्क समस्या के लिए एक नया अवशिष्ट–आधारित बिंदुवार एक पश्चवर्ती त्रुटि अनुमानक विकसित करते हैं। विश्लेषण में, हम संपर्क क्षेत्र पर निरंतर टुकड़े–टुकड़े द्विघात परिमित तत्व स्थान और निरंतर टुकड़े–टुकड़े रैखिक गुणक के साथ काम करते हैं। अध्याय 5 में, हम दीर्घवृत्त बाधा समस्या के लिए द्विघात परिमित तत्व विधि का बिंदुवार पश्च त्रुटि विश्लेषण प्राप्त करते हैं। प्रस्तावित पश्च त्रुटि अनुमानक की विश्वसनीयता और दक्षता पर चर्चा की गई है। नियमित ग्रीन के कार्य की सीमा और असतत लैग्रेंज गुणक का संकेत

विश्लेषण में महत्वपूर्ण भूमिका निभाता है। अध्याय 6 में, हम बाधा समस्या के लिए द्विघात डीजी विधियों के सर्वोच्च मानदंड में एक पश्चवर्ती त्रुटि विश्लेषण करते हैं। हम दो असतत सेटों को परिभाषित करते हैं (गद्दाम, गुड़ी और कामना [44] से प्रेरित) एक अभिन्न बाधाओं के साथ और दूसरा चतुर्भुज बिंदुओं पर नोडल बाधाओं के साथ, और प्रस्तावित पश्चगामी त्रुटि अनुमानक की बिंदुवार विश्वसनीयता और दक्षता पर चर्चा करते हैं। विश्लेषण में, हम डीजी परिमित तत्व स्थान को मानक अनुरूप परिमित तत्व स्थान पर स्थानांतरित करने के लिए एक रैखिक औसत कार्य को नियोजित करते हैं और पॉइसन समस्या के ग्रीन के कार्य पर तेज सीमा का फायदा उठाते हैं। अध्याय 2 से अध्याय 6 में कई संख्यात्मक प्रयोग प्रस्तुत किए गए हैं, जो विश्लेषण में प्राप्त अभिसरण के सैद्धांतिक क्रम को संख्यात्मक रूप से दर्शाते हैं। अध्याय 7 में, हम किए गए कार्यों का सार प्रस्तुत करते हैं और भविष्य की जांच के लिए गुंजाइश के साथ संभावित विस्तार प्रस्तावित हैं।

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Symbol	**Meaning**
$\mathcal{T}_h$	a regular triangulation of Ω,
T	a simplex of $\mathcal{T}_h$,
$\lvert T\rvert$	volume of an element T,
Γ^D	The Dirichlet boundary,
Γ^N	The Neumann boundary,
Γ^C	The contact boundary,
p	a node of an element T,
$\mathcal{V}_h$	set of all vertices in $\mathcal{T}_h$,
$\mathcal{V}_T$	set of vertices of the simplex T,
$\mathcal{V}_h^i$	set of all vertices in $\mathcal{T}_h$ that are in Ω,
$\mathcal{V}_h^D$	set of all vertices on Γ^D,
$\mathcal{V}_h^N$	set of all vertices on Γ^N,
$\mathcal{V}_h^{\bar{N}}$	set of all vertices on the closure of Γ^N,
$\mathcal{V}_h^C$	set of all vertices on the potential contact boundary ,
$\mathcal{V}_h^0$	set of all vertices in $\mathcal{T}_h$ that are in $\mathcal{V}_h \setminus \mathcal{V}_h^D$,
$\mathcal{V}_h^e$	set of all vertices lying on edge e,
$\mathcal{E}_h$	set of all edges or faces of $\mathcal{T}_h$,
$\mathcal{E}_h^i$	set of all interior edges or faces of $\mathcal{T}_h$,
$\mathcal{E}_p$	set of all faces connected to a given vertex p,
$\mathcal{M}_h^e$	set of all midpoints lying on edge e,
$\mathcal{M}_h^i$	set of midpoints of interior edges or faces in $\mathcal{T}_h$,
$\mathcal{M}_h^C$	set of all midpoints lying on the potential contact boundary,
$\mathcal{M}_h^D$	set of all midpoints lying on Γ^D,

$\mathcal{M}_h^0$	set of all midpoints in $\mathcal{T}_h$ that are in $\mathcal{M}_h \setminus \mathcal{M}_h^D$,
$\mathcal{M}_T$	set of midpoints of all the edges of the simplex T,
$P_k(T)$	the space of polynomials of degree atmost k over element T,
h_e	length of an edge or face $e \in \mathcal{E}_h$,
ω_p	union of all elements sharing the node p,
ω_e	$\bigcup_{e \subset \partial T} T$ for any face e,
h_p	diameter of ω_p,
$\gamma_{p,I}$	union of all faces in the interior of ω_p not including the boundary of ω_p,
$\gamma_{p,C}$	$\Gamma^C \cap \partial\omega_p$,
$\gamma_{p,D}$	$\Gamma^D \cap \partial\omega_p$,
$\gamma_{p,N}$	$\Gamma^N \cap \partial\omega_p$,
$\|\|\cdot\|\|_{m,p,D}$	$\|\|\cdot\|\|_{W^{m,p}(D)}$ where $m \in \mathbb{Z}$ and $1 \leq p \leq \infty$ and D $\subseteq \Omega$,
$\|\|\cdot\|\|_{0,\infty}$	$\|\|\cdot\|\|_{L^\infty(\Omega)}$,
$\|\|\cdot\|\|_{0,\infty,A}$	$\|\|\cdot\|\|_{L^\infty(A)}$, where $A \subseteq \Omega$,
$\|\cdot\|_{m,p,A}$	The associated semi norm on $W^{m,p}(A)$ where $A \subseteq \Omega$,
$\delta_{i,j}$	The Kronecker's Delta i.e., $\delta_{i,j} = 1$ if $i = j$ and 0 elsewhere,
$\boldsymbol{v}'$	The transpose of vector valued function $\boldsymbol{v}$,
$L^p(\Omega)$	Lebesgue spaces, $1 \leq p \leq \infty$,
$W^{k,p}(\Omega)$	Sobolev spaces, $1 \leq p \leq \infty$,
$H^k(\Omega),\ H_0^k(\Omega)$	Sobolev spaces,
$C^k(\Omega)$	The space of k times continuously differentiable functions,
$C^{k,\theta}(\Omega)$	Hölder spaces with exponent $\theta \in (0,1)$,
$D(\Omega)$	The space of infinitely differentiable functions having compact support in Ω,
C	Generic positive constant, independent of perturbation and mesh parameters,
$X \lesssim Y$	$X \leq CY$ for some positive constant C, which is independent of the mesh size h,
$L^2_{loc}(\Omega)$	$\{f : \Omega \to \mathbb{R} \mid f\|_K \in L^2(\Omega), K \text{ is compact subset of } \Omega\}$
$C_c^k(\Omega)$	The space of k times continuously differentiable functions with compact support in Ω,
$BV(\Omega)$	The space of functions with bounded variation on Ω,
$\|Dg\|_{var}$	$\sup\{\int_\Omega g\ div(\phi)\ dx : \phi \in C_c^1(\Omega), \|\|\phi\|\|_{L^\infty(\Omega)} \leq 1\}$,

$meas(A)$	Measure of the set $A \subseteq \bar{\Omega}$,
$supp(f)$	Support of function f,
$\|a\|$	Absolute value of a, $a \in \mathbb{R}$.