ENTROPY STABLE SCHEMES FOR CONSERVATIVE AND NON-CONSERVATIVE HYPERBOLIC SYSTEMS

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by

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Dedicated to my mother-in-law Mrs. Vidhya Devi and father-in-law Mr. Kamal Singh.

Certificate

This is to certify that the thesis entitled **Entropy stable schemes for conservative and non-conservative hyperbolic systems** submitted by **Mrs. Anshu Yadav** to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

Harish

New Delhi
July 2024

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Abstract

This thesis focuses on two main objectives: the development of high-order entropy-stable discontinuous Galerkin schemes for conservation laws and high-order entropy-stable finite difference schemes for hyperbolic systems involving non-conservative products. The proposed schemes ensure nonlinear stability in terms of entropy stability. Moreover, the design of an entropy-stable finite difference scheme for non-conservative hyperbolic systems is highly nontrivial and the first work in this direction.

The first part of the thesis addresses the design of an entropy-stable discontinuous Galerkin scheme for the ten-moment Gaussian closure equations. In various applications involving plasma flows, the fluid components are commonly modeled using the Euler equations of compressible flows. These equations are derived by taking velocity moments of the Boltzmann equation and assuming local thermodynamic equilibrium. However, the assumption of local thermodynamic equilibrium is not valid for specific applications, particularly those related to plasma flows. In such cases, it becomes necessary to account for the anisotropic nature of the pressure. To address this issue, Levermore *et. al.* proposed the Ten-Moment equations model. This model results in a hyperbolic system of conservation laws, where the pressure is described using a symmetric tensor. The resulting system is non-linear and hyperbolic, leading to the presence of discontinuities in the solution of the corresponding Cauchy problem, even with smooth initial data. Consequently, weak solutions are considered, and an additional criterion in the form of entropy stability is imposed to exclude physically irrelevant solutions. Due to the non-linearity of the flux function, the existence of theoretical solutions is generally not possible for most Cauchy problems. Therefore, computational methods are used for practical applications, highlighting the significance of the first contribution made by this thesis.

The first contribution of the thesis is to design high-order discontinuous Galerkin entropy stable schemes for ten-moment Gaussian closure equations based on the suitable quadrature

rules. The key components of the schemes are the use of an entropy conservative numerical flux in each cell and an appropriate entropy stable numerical flux at the cell edges. These fluxes are then used in the entropy-stable DG framework to obtain entropy stability of the semi-discrete schemes. We also extend these schemes to a source term that models plasma laser interaction. For time discretization, we use strong stability-preserving methods. The proposed schemes are then tested on several test cases to demonstrate stability, accuracy, and robustness.

The second part of the thesis addresses the design of an entropy-stable finite difference scheme for non-conservative hyperbolic systems. The fundamental concept is to reformulate the system so that the non-conservative terms do not contribute to the evolution of entropy. Then, we first construct an entropy conservative scheme following the ideas of Tadmor and then add dissipative terms that lead to an entropy inequality. We then establish fully discrete schemes using the high-order accurate Strong Stability Preserving Runge Kutta (SSP-RK) time-stepping methods. The proposed scheme is then applied to two systems, the shear shallow water model (SSW) and the Chew, Goldberger & Low system (CGL), demonstrating its effectiveness in handling various non-conservative hyperbolic systems. Consequently, this represents the next significant contribution of the thesis.

The second contribution of the thesis involves the development of a high-order entropy stable finite difference scheme for the shear shallow water (SSW) model. The SSW model extends the classical shallow water model by incorporating the effects of vertical shear. It comprises of six non-linear hyperbolic partial differential equations (PDEs) with non-conservative products. While the SSW model falls under the category of non-conservative hyperbolic systems, it exhibits a unique structure where both conservative and non-conservative terms are present. The conservative terms have a similar structure to the ten-moment equations of gas dynamics. To design an entropy-stable scheme for the SSW model, the thesis exploits the conservation form of the equations and the special structure of the non-conservative terms, which do not contribute to the entropy. The scheme is initially developed in one dimension and then extended to two dimensions using logically rectangular meshes. The stability and accuracy of the proposed scheme are thoroughly demonstrated on many test cases in one and two dimensions.

The third contribution of the thesis involves designing a high-order entropy stable finite difference scheme for the Chew, Goldberger & Low (CGL) equations. It is a set of non-linear, non-conservative hyperbolic PDEs modeling anisotropic plasma flows. In this contribution, we first present the entropy analysis for the weak solutions. The key idea in designing the numerical scheme is to rewrite the CGL equations such that the non-conservative terms do not contribute

to the entropy equations. The conservative part of the rewritten equations is very similar to the magnetohydrodynamics (MHD) equations. We then symmetrize the conservative part by following Godunov's symmetrization process for MHD. The resulting equations are then discretized by designing entropy conservative numerical flux and entropy diffusion operator based on the entropy scaled eigenvectors of the conservative part. We then prove the semi-discrete entropy stability of the schemes for CGL equations. The schemes are then tested using several test problems derived from the corresponding MHD test problems.

सार

यह थीसिस दो मुख्य उद्देश्यों पर केंद्रित है: संरक्षण कानूनों के लिए उच्च-क्रम एन्ट्रॉपी स्थिर असंतत गैलेर्किन योजनाओं का विकास और गैर-रूढ़िवादी उत्पादों को शामिल करने वाले हाइपरबोलिक सिस्टम के लिए उच्च-क्रम एन्ट्रॉपी स्थिर परिमित अंतर योजनाओं का विकास। प्रस्तावित योजनाएं एन्ट्रापी स्थिरता के संदर्भ में गैर-रैखिक स्थिरता सुनिश्चित करती हैं। इसके अलावा, गैर-रूढ़िवादी हाइपरबोलिक प्रणालियों के लिए एन्ट्रापी स्थिर परिमित अंतर योजना का डिज़ाइन अत्यधिक गैर-तुच्छ है और इस दिशा में पहला काम है।

थीसिस का पहला भाग दस-क्षण गॉसियन क्लोजर समीकरणों के लिए एन्ट्रापी स्थिर असंतत गैलेर्किन योजना के डिजाइन को संबोधित करता है। प्लाज्मा प्रवाह से जुड़े विभिन्न अनुप्रयोगों में, द्रव घटकों को आमतौर पर संपीड़ित प्रवाह के यूलर समीकरणों का उपयोग करके मॉडल किया जाता है। ये समीकरण बोल्ट्ज़मैन समीकरण के वेग क्षणों को लेकर और स्थानीय थर्मोडायनामिक संतुलन मानकर प्राप्त किए गए हैं।हालाँकि, कुछ अनुप्रयोगों के लिए, विशेष रूप से प्लाज्मा प्रवाह से संबंधित, स्थानीय थर्मोडायनामिक संतुलन की धारणा मान्य नहीं है। ऐसे मामलों में, दबाव की अनिसोट्रोपिक प्रकृति को ध्यान में रखना आवश्यक हो जाता है। इस समस्या का समाधान करने के लिए, लीवरमोर ने दस-मोमेंट समीकरण मॉडल प्रस्तावित किया। इस मॉडल के परिणामस्वरूप संरक्षण कानूनों की एक अतिपरवलयिक प्रणाली बनती है, जहां एक सममित टेंसर का उपयोग करके दबाव का वर्णन किया जाता है। परिणामी प्रणाली गैर-रैखिक और अतिशयोक्तिपूर्ण है, जिससे संबंधित कॉची समस्या के समाधान में सहज प्रारंभिक डेटा के साथ भी असंतोष की उपस्थिति होती है। नतीजतन, कमजोर समाधानों पर विचार किया जाता है, और भौतिक रूप से अप्रासंगिक समाधानों को बाहर करने के लिए एन्ट्रापी स्थिरता के रूप में एक अतिरिक्त मानदंड लगाया जाता है। फ्लक्स फ़ंक्शन की गैर-रैखिकता के कारण, अधिकांश कॉची समस्याओं के लिए सैद्धांतिक समाधान का अस्तित्व आम तौर पर संभव नहीं है। इसलिए, व्यावहारिक अनुप्रयोगों के लिए कम्प्यूटेशनल तरीकों का उपयोग किया जाता है, जो इस थीसिस द्वारा किए गए पहले योगदान के महत्व पर प्रकाश डालता है।

थीसिस का पहला योगदान उपयुक्त चतुर्भुज नियमों के आधार पर, दस-क्षण गॉसियन क्लोजर समीकरणों के लिए उच्च क्रम की असंतत गैलेरकिन एन्ट्रापी स्थिर योजनाओं को डिजाइन करना है। योजनाओं के प्रमुख घटक प्रत्येक कोशिका में एक एन्ट्रापी रूढ़िवादी संख्यात्मक प्रवाह और सेल किनारों पर एक उपयुक्त एन्ट्रापी स्थिर संख्यात्मक प्रवाह का उपयोग होते हैं।फिर इन फ्लक्स का उपयोग अर्ध-असतत योजनाओं की एन्ट्रापी स्थिरता प्राप्त करने के लिए एन्ट्रापी स्थिर डीजी ढांचे में किया जाता है। हम इन योजनाओं को एक स्रोत शब्द तक भी विस्तारित करते हैं जो प्लाज्मा लेजर इंटरैक्शन को मॉडल

करता है। समय विवेकीकरण के लिए, हम मजबूत स्थिरता संरक्षण विधियों का उपयोग करते हैं। स्थिरता, सटीकता और मजबूती प्रदर्शित करने के लिए प्रस्तावित योजनाओं का कई परीक्षण मामलों पर परीक्षण किया जाता है।

थीसिस का दूसरा भाग गैर-रूढ़िवादी हाइपरबोलिक प्रणालियों के लिए एन्ट्रापी स्थिर परिमित अंतर योजनाओं के डिजाइन को संबोधित करता है। मूल अवधारणा प्रणाली को इस तरह से पुनर्गठित करना है कि गैर-रूढ़िवादी शब्द एन्ट्रापी विकास में योगदान न करें। फिर हम पहले टैडमोर के विचारों का अनुसरण करते हुए एक एन्ट्रापी रूढ़िवादी योजना का निर्माण करते हैं और फिर विघटनकारी शब्द जोड़ते हैं जो एन्ट्रापी असमानता को जन्म देते हैं।फिर हम उच्च-क्रम सटीक स्ट्रॉन्ग स्टेबिलिटी प्रिजर्विंग रन्ज कुट्टा (एसएसपी-आरके) टाइम-स्टेपिंग विधियों का उपयोग करके पूरी तरह से अलग योजनाएं स्थापित करते हैं। प्रस्तावित योजना को फिर दो अलग-अलग प्रणालियों, कतरनी उथले पानी मॉडल (एसएसडब्ल्यू) और च्यू, गोल्बर्गर पर लागू किया जाता है।

थीसिस के दूसरे योगदान में कतरनी उथले पानी (एसएसडब्ल्यू) मॉडल के लिए एक उच्च-क्रम एन्ट्रापी स्थिर परिमित अंतर योजना का विकास शामिल है। एसएसडब्ल्यू मॉडल ऊर्ध्वाधर कतरनी के प्रभावों को शामिल करके शास्त्रीय उथले पानी मॉडल का विस्तार करता है। इसमें गैर-रूढ़िवादी उत्पादों के साथ छह गैर-रेखीय हाइपरबोलिक आंशिक अंतर समीकरण (पीडीई) शामिल हैं।जबकि एसएसडब्ल्यू मॉडल गैर-रूढ़िवादी हाइपरबोलिक प्रणालियों की श्रेणी में आता है, यह एक अनूठी संरचना प्रदर्शित करता है जहां रूढ़िवादी और गैर-रूढ़िवादी दोनों शब्द मौजूद हैं। रूढ़िवादी शब्दों की संरचना गैस गतिकी के दस-क्षण समीकरणों के समान होती है। एसएसडब्ल्यू मॉडल के लिए एक एन्ट्रापी स्थिर योजना को डिजाइन करने के लिए, थीसिस समीकरणों के संरक्षण रूप के साथ-साथ गैर-रूढ़िवादी शब्दों की विशेष संरचना का उपयोग करती है, जो एन्ट्रापी में योगदान नहीं करती है। योजना को प्रारंभ में एक आयाम में विकसित किया गया है और फिर तार्किक रूप से आयताकार जालों का उपयोग करके दो आयामों तक विस्तारित किया गया है। प्रस्तावित योजना की स्थिरता और सटीकता को एक और दो आयामों में कई परीक्षण मामलों पर पूरी तरह से प्रदर्शित किया गया है।

थीसिस के तीसरे योगदान में च्यू, गोल्डबर्गर के लिए एक उच्च-क्रम एन्ट्रापी स्थिर परिमित अंतर योजना का डिज़ाइन शामिल है। संख्यात्मक योजना को डिजाइन करने में मुख्य विचार सीजीएल समीकरणों को फिर से लिखना है ताकि गैर-रूढ़िवादी शब्द एन्ट्रापी समीकरणों में योगदान न करें। पुनर्लिखित समीकरणों का रूढ़िवादी हिस्सा मैग्नेटोहाइड्रोडायनामिक्स (एमएचडी) समीकरणों के समान है। फिर हम एमएचडी के लिए गोडुनोव की समरूपता प्रक्रिया का पालन करके रूढ़िवादी भाग को सममित करते हैं।इसके बाद परिणामी समीकरणों को रूढ़िवादी भाग के एन्ट्रॉपी स्केल किए गए ईजेनवेक्टरों के आधार पर एन्ट्रॉपी रूढ़िवादी संख्यात्मक प्रवाह और एन्ट्रॉपी प्रसार ऑपरेटर को डिजाइन करके अलग किया जाता है। फिर हम सीजीएल समीकरणों के लिए योजनाओं की अर्ध-असतत एन्ट्रापी स्थिरता साबित करते हैं। फिर संबंधित एमएचडी परीक्षण समस्याओं से प्राप्त कई परीक्षण समस्याओं का उपयोग करके योजनाओं का परीक्षण किया जाता है।

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List of Symbols

$\mathbb{R}$	Set of Real numbers
$\mathbb{Z}$	Set of Integers
$\mathbb{N}$	Set of Natural numbers
$\mathbb{R}^n$	n-dimensional Cartesian Product of $\mathbb{R}$
$\in$	Belongs to
$\subset$	Subset
$\forall$	For all
$(\cdot)^\top$	Transpose of a vector (or a matrix)
$\nabla \cdot \mathbf{F}$	Gradient of a vector $\mathbf{F}$
$\Delta \mathbf{U}$	Hessian of a vector $\mathbf{U}$
$\mathbf{C}^1(D)$	Space of differentiable functions whose derivative is continuous
$\mathbf{L}^\infty_{loc}(D)$	Space of Locally almost everywhere bounded integrable function over a domain D
$\mathbf{C}^\infty_0(D)$	Space of continuously differentiable functions with compact support
$\mathbf{U}$	Vector of conservative variables
$\mathbf{W}$	Vector of primitive variables
$\mathbf{f}^j(\mathbf{U})$	Exact continuous fluxes
$\mathbf{\Lambda}^j$	Set of eigenvalues for the jacobian matrix $\frac{\partial \mathbf{f}^j}{\partial \mathbf{U}}$
$\mathbf{R}^j$	The matrix of right eigenvectors for the jacobian matrix $\dfrac{\partial \mathbf{f}^j}{\partial \mathbf{U}}$
$(\mathcal{U}(\mathbf{U}), \mathcal{F}^j(\mathbf{U}))$	Entropy-entropy flux pair
$\mathbf{V}(\mathbf{U})$	Entropy variable

Motivation

Partial Differential Equations (PDEs) are crucial tools for modeling in various fields. They are used because most physical laws describe changes in quantities, and PDEs make expressing these changes manageable. Specifically, we are interested in first-order Hyperbolic PDEs.

Hyperbolic PDEs appear in several interesting physical problems, e.g., oceanography(shallow water equations), aerodynamics (Euler compressible fluid flow equation), plasma physics (Equations of magnetohydrodynamics (MHD)) and structural mechanics (non-linear elasticity). In these systems, the equations are often formulated in the context of conservation laws, where the rate of change of conservative quantities is balanced with the flux through boundaries and source terms. This means that the change in these quantities inside a given region is equal to the flow of these quantities through the boundaries of that region, and sources or sinks of these quantities present inside the region. However, not all systems can be written in the conservative form. Some systems need to be reformulated in an "almost conservative" form, where the equations involve both flux and non-conservative terms.

Often, hyperbolic systems have non-linear flux. This leads to the loss of regularity of solutions, even if the initial data is smooth. Hence, smooth solutions are possible only for special initial conditions, and often, they are not helpful for practical problems. So, we consider the weak formulation of the equations. However, the class of weak solutions is large and contains physically irrelevant solution to the problem. So, an additional criterion is required to pick out the physically relevant solution. This is achieved by enforcing the entropy condition. Although uniqueness is not guaranteed even with the entropy condition, it provides a non-linear stability estimate, which is highly desirable.

In general, finding analytical solutions to the hyperbolic systems is not possible; therefore, numerical methods are used to obtain approximate solutions. For hyperbolic PDEs, numerical methods are based on Finite Volume, Finite Difference ([1, 2]), and Discontinuous Galerkin methods ([3]). The solution produced by these numerical methods should satisfy a discrete version of the entropy condition.

The development of entropy stable schemes began with the work of Tadmor in several articles (see [4–8]). A critical step in designing entropy stable schemes is the construction of high-order accurate entropy conservative fluxes. To accomplish this, the notion of entropy variables and entropy potentials are introduced. The development of second-order entropy conservative flux is then based on the theorem by *Tadmor* (see [6, Theorem 4.1]). High-order accurate entropy conservative fluxes are obtained following the procedure in [9]. Here, high-order entropy conservative numerical fluxes are constructed using specific linear combinations of the second-order accurate conservative fluxes.

The entropy conservative schemes will produce high-frequency oscillations near the shocks as they do not dissipate entropy at the shocks. Therefore, using the approach given in [6], the Roe diffusion operators are introduced to construct entropy stable schemes. However, the diffusion term involves only first-order accurate jumps of the entropy variables. Hence, the obtained entropy stable schemes are only first-order accurate even though the conservative flux is high-order accurate. To overcome this, following the work in [10], a *sign-preserving* reconstruction procedure is used to obtain high-order accurate diffusion operators. The reconstruction procedures are applied to scaled entropy variables. As the *min-mod* reconstruction procedure preserves *sign-property*, a spatially second-order accurate scheme is obtained by using the *min-mod* reconstruction (see [10, 11]). For higher than second-order accurate schemes, high-order accurate conservative flux along with the corresponding high-order *ENO* reconstruction (see [12]) of the scaled entropy variables are used. The resulting semi-discrete schemes are then shown to be entropy stable. For the time update, Strong Stability Preserving Runge Kutta (SSP-RK) methods (see [13]) are used. These schemes have been successfully applied to approximate various hyperbolic systems, such as Euler equation of gas dynamics [14], Magnetohydrodynamics (MHD) equations [15], Relativistic Hydrodynamic equations (RHD) [16] and equation of ten-moment gaussian closure [17].

Meanwhile, Cockburn et al. have developed discontinuous Galerkin (DG) schemes ([18–20]), which are very popular. The entropy-stable DG schemes are developed by Carpenter, Gassner, Fisher, and others ([21–26]). More recently, a general framework for constructing an entropy-stable DG scheme for hyperbolic conservation laws was proposed by Chen and Shu in [27] and applied to Euler equations for compressible flows.

To simplify the discussion, firstly, the one-dimensional case can be considered. Extension to higher dimensions is based on the tensor product in each direction. The discrete finite element space is defined as a collection of piecewise continuous polynomials. For k^{th} order accurate schemes, polynomials of degree at most $k-1$ are used.

In DG schemes, we search for a solution in the finite element space that satisfies a weak formulation of the conservation law (see [27]). The interface flux used in the weak formulation can be obtained either using an exact Riemann solver or an approximate Riemann solver. The integrals in the weak formulation are evaluated using the Legendre-Gauss-Lobatto quadrature rule (see [28]) over the reference interval $[-1, 1]$. The Lagrangian basis defines the nodal basis for the finite element space. Using the SBP property (see [27]), the weak formulation gives rise to the DG schemes. In general, the schemes are not entropy stable. Therefore, the schemes need to be modified. To proceed, one can recall the notion of entropy variables and entropy potentials. The entropy conservative flux obtained using the *Tadmor's* theorem [6], can be incorporated into the DG schemes to obtain conservative schemes, k^{th} order accurate, and entropy conservative in the discrete sense. The obtained schemes are only entropy conservative and do not dissipate entropy at the shocks. To design an entropy-stable scheme, the interface flux is considered entropy-stable (see [27]). An expected entropy stable flux, used often, is the Lax-Friedrich flux (see [11],[29]). The framework is easily extendable to multi-dimensional systems of hyperbolic conservation laws (or balance laws).

For the time update, we use explicit SSP-RK methods. Sometimes, the solution obtained using the entropy-stable DG schemes may produce spurious high-frequency oscillations. Hence, we use slope limiters. These limiters can be found in [30],[31]. Another concern is the positivity of the solutions during the time evolution. We may require the Physically Constraints Preserving (PCP) limiter to ensure the solution is in the physical domain. Some of the popular PCP limiters for various hyperbolic systems can be found in [32],[33],[34].

In the subsequent chapters, we will first present the frameworks mentioned above. We will then apply these frameworks to three specific systems: the Ten-moment Gaussian closure equations, shear shallow water model equations, and Chew, Goldberger, & Low (CGL) equations for plasma flow equations.

The first contribution of the thesis is to design high-order entropy stable nodal DG schemes for the Ten-moment Gaussian closure equations, which models the rarefied gas flows. The local thermodynamic equilibrium assumption does not hold in several fluid and plasma flow applications. Due to this, the simulations of such flows using the scalar description of the pressure (used most commonly in Euler equations of compressible fluid flows) need to be revised.

A more accurate description of tensorial pressure (and hence temperature) is needed. One such model is the ten-moment Gaussian closure equations. It is one of the simplest fluid models considering pressure as a tensor. The ten-moment Gaussian closure equations are a hyperbolic system of Balance laws with non-linear flux.

Due to the non-linearity of the flux function, the theoretical existence of the solutions is not possible for most of the Cauchy problems. Hence, computational methods are used for most of the applications.

In this contribution, we use the entropy conservative numerical flux presented in [17] for ten-moment Gaussian closure equations. We first formulate the DG schemes for one-dimensional ten-moment equations. The DG schemes are then modified using entropy conservative flux of the ten-moment Gaussian closure equations to obtain the entropy conservation. This is then coupled with Lax-Friedrich flux (see [11, 29]) to obtain cell entropy stability of the semi-discrete schemes. We then generalize the one-dimensional schemes to the two-dimensions We use explicit SSP-RK methods for the time update. We present second, third, and fourth-order schemes.

As discussed, the solution obtained using the DG schemes may require a limiter treatment to give oscillation-free profiles. For the one-dimensional problems, we use the slope limiter given in [30], and for the two-dimensional case, we use the approach of [31] to define the two-dimensional slope limiter. In both directions, we compute the solutions using $M = 10$. We use the bound preserving limiter of [33] to keep the solution in the physical domain.

The proposed schemes then tested on various test cases, including several Riemann problems.

In the second contribution of this thesis, we design high-order finite-difference schemes for the shear shallow water (SSW) equations, which are entropy stable. It is a hyperbolic system consisting of six non-linear PDEs with non-conservative products. The system of equations describing multi-dimensional shear shallow water (SSW) flows was derived by Teshukov. This system approximates shallow water flows by including the effects of vertical shear, which are neglected in the classical shallow water (Saint-Venant) model. The resulting system of equations closely resembles the ten-moment Gaussian closure model of gas dynamics, except for some additional terms arising from gravitational effects. In particular, the entropy function of the two models is the same since the non-conservative terms in the SSW model, which are purely due to gravitational effects, do not contribute to the entropy equation. Being a non-conservative hyperbolic system, the design of numerical methods for the SSW model is challenging since the notion of a weak solution requires choosing a path that is usually unknown. The correct path depends on the physical regularization mechanism, and even when the correct path is known, constructing a numerical scheme that converges to the weak solution is challenging since the solution is sensitive to the numerical viscosity. A few methods have been developed for the SSW model, following HLL-type ideas in ([35–37]).

In this contribution, we exploit the special structure of the non-conservative terms, which do not contribute to the entropy. The primary idea is to first construct an entropy conservative scheme following the ideas of Tadmor and then add dissipative terms [14, 38] that leads to an entropy inequality. For conservative systems, constructing the entropy conservative scheme is based on finding a central numerical flux that satisfies a certain jump condition [8]. Since the non-conservative terms do not contribute to the entropy, the ideas from conservative systems can be used to construct an entropy conservative scheme. The high-order accuracy is achieved by following the ideas in LeFloch [9].

The scheme is first developed in one dimension and then extended to two dimensions on rectangular meshes. The stability and accuracy of the proposed schemes are demonstrated on many test cases in one and two dimensions in comparison to some exact solutions and the computation of roll waves that are observed in some experimental studies.

In this work, we also studied the relationship between the existence of entropy pair and symmetrizability for the general non-conservative hyperbolic systems. For the conservation laws, it is well known that the existence of a convex entropy function and the symmetrization of the equations (see [39], Theorem 3.2) are closely related. However, we have proved that this property does not hold for general non-conservative systems; for the SSW model, we have a convex entropy function and an entropy conservation law for smooth solutions, but the equations cannot be symmetrized. The failure to symmetrize is due to the non-conservative terms related to gravitational effects, but since they do not contribute to the entropy equation, we still have an entropy equation satisfied by smooth solutions.

The thesis's third contribution is to design high-order entropy stable finite difference schemes for the Chew Goldberger & Low equations. These equations were first proposed in [40] and are the simplest model to describe anisotropic plasma flows. In addition to the mass and momentum conservation equations, the fluid is described using two scalar pressure components (one parallel to the magnetic field and another perpendicular to the magnetic field). Furthermore, the evolution of the magnetic field is modeled via an induction equation. The CGL equations are a set of non-conservative hyperbolic systems of PDEs. Due to this, the design of the numerical schemes for CGL equations is challenging. The weak solution formulations for these equations require choosing an appropriate path, which is often unknown. Furthermore, the numerical solutions are sensitive to the numerical viscosity. A linear path is often considered to design approximated Riemann solvers ([41, 42]).

In this contribution, we consider entropy stable discretization of the equations following the ideas in [8, 10, 14, 15, 38, 43]. We first present the entropy analysis of the system at the continuous level. Following the idea of [43], we reformulate the CGL equations so that the non-conservative terms do not contribute to the evolution of entropy. We realized that the conservative flux of the reformulated equations is very similar to the MHD flux. Furthermore, the entropy equation for the conservative part depends on the divergence of the magnetic field. Hence, we use Godunov's symmetrization for the conservative equations following [15]. We then design entropy conservative numerical flux for the conservative part of equations. Following [10], we also design a higher-order numerical diffusion operator based on the entropy-scaled eigenvectors and sign-preserving reconstruction process. Finally, combining the entropy-stable discretization of the conservative part with a non-conservative part, we prove the semi-discrete entropy stability of the proposed scheme.

The schemes are then tested using several test problems derived from the corresponding MHD test problems.

The thesis consists of six Chapters:

In **Chapter 1**, we present a brief introduction to the hyperbolic systems, covering both conservative and non-conservative formulations. We have divided the chapter into two sections. The first section starts by defining the notion of a Cauchy problem, followed by the definition of weak solutions for conservation laws. To choose the physically relevant solution, we introduce the notion of entropy condition. We end this section by giving an example of the Shallow water model, a non-linear hyperbolic conservation law. In the second section, we present the theoretical aspects of non-conservative hyperbolic systems. In **Chapter 2**, we describe high-order entropy stable schemes for general hyperbolic conservation laws. It covers both the finite-difference and discontinuous Galerkin frameworks. As the first original contribution, in **Chapter 3**, we design entropy stable discontinuous Galerkin schemes for the ten-moment Gaussian closure equations. The work is based on the framework presented in Chapter 2. As the second original contribution of the thesis, in **Chapter 4**, we design arbitrarily high-order accurate finite-difference entropy stable schemes for the Shear Shallow water model equations. In **Chapter 5**, we describe the Chew Goldberger & Low equations and discuss the hyperbolicity, entropy framework, and symmetrizability of the system. We then reformulated the system, such that the non-conservative terms do not affect in entropy evolution. In another key contribution of this thesis, we design arbitrarily high-order accurate finite-difference entropy stable schemes for the CGL equations in **Chapter 6**.