NON-COMPACT VARIATIONAL PROBLEMS IN THE HYPERBOLIC SPACE

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Non-Compact Variational Problems in the Hyperbolic Space

by

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Dedicated to my grandfather.

Certificate

This is to certify that the thesis entitled **Non-Compact Variational Problems in the Hyperbolic Space** submitted by **Ms. Diksha Gupta** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.
The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

Non-compact variational elliptic problems are an intriguing and demanding domain of research within the realm of PDEs, where standard approaches often encounter substantial challenges. The study of equations involving critical nonlinearities, singularities, or unbounded domains may give rise to such noncompactness. In this thesis, we aim to study the Brezis-Nirenberg type problem in the hyperbolic space with suitable perturbations. The lack of compactness in such problems stems from either the infinite volume of the domain or the critical exponent ($2^* = \frac{2N}{N-2}$) in sense of the Poincaré-Sobolev inequality (2.3) established by Mancini-Sandeep [118].

Understanding the behaviour of solutions to these equations with Sobolev critical exponent is crucial because they arise in many different areas of physics and engineering, including nonlinear optics, Bose-Einstein condensates, fluid mechanics, and more. In addition, studying these equations is an active area of research in mathematics, with many challenging open problems remaining.

The principle of concentration compactness (see [111, 112]) categorizes the failure of compactness into three main categories: *concentration*, where mass gathers onto a small set; *vanishing*, where mass dissipates to infinity; and *dichotomy*, where mass splits into two diverging masses. Thus unlike compact settings, where sequences of functions exhibit well-behaved convergence properties, non-compactness introduces subtleties that demand novel approaches and sophisticated techniques for analysis such as concentration-compactness principles, mountain pass theorems, and Palais–Smale condition (see [152]).

In the esteemed work authored by Mancini and Sandeep [118], they demonstrated that the equation (1.8) possesses a unique positive weak solution $\mathcal{U} := \mathcal{U}_{N,\lambda}$ (upto to hyperbolic isometries), which also serves as an extremal of the Poincaré-Sobolev inequality (2.3). Motivated by this, we delved into the examination of the perturbations of the problem (1.8) in the subcritical case, handling the cases when $f \equiv 0$, and when $f \not\equiv 0$. Furthermore, in our pursuit of exploring critical exponent problems, we initiated our study with an analysis of the critical perturbations problem. In the subcritical case, the variational problem lacks compactness because of the *hyperbolic translations* (refer [143]).

However, for critical exponent problems in the hyperbolic space, loss of compactness can also happen along concentration of Aubin-Talenti bubble (locally). To address the critical exponent problem, we established the Palais-Smale decomposition. Leveraging this global compactness result, we derived several existence and multiplicity results when $f \not\equiv 0$. Subsequently, we explored the purely critical exponent problem, that is, $f \equiv 0$, utilizing the concept of critical points at infinity introduced by A. Bahri. This approach is more intricate and necessitated the development of several delicate estimates. Finally, in view of the Hardy-Littlewood-Sobolev inequality in the hyperbolic space, we investigated the existence and qualitative properties of ground state solutions to a nonlinear Choquard equation.

सार

गैर-संपीडनीय विविक्त प्रकार की एलिप्टिक समस्याएँ आंशिक अवकल समीकरणों के क्षेत्र में एक अत्यंत रोचक और चुनौतीपूर्ण शोध विषय हैं, जहाँ पारंपरिक तरीकों से समस्याएँ हल करना कठिन हो जाता है। जब समीकरणों में क्रांतिक रेखिकताएँ विषमताएँ , या अनबाउंडेड डोमेन सम्मिलित होते हैं, तब ऐसी गैर-संपीडनीय स्थितियाँ उत्पन्न हो सकती हैं। इस शोध कार्य में, हमने उपयुक्त व्यवधानों के साथ हाइपरबोलिक स्थान में ब्रेज़ी-निरेंबर्ग प्रकार की समस्या का अध्ययन किया है। ऐसी समस्याओं में संपीडन की कमी का कारण या तो डोमेन की अनंत मात्रा होती है या फिर पॉइंकारे-सोबोलेव असमानता के अनुसार मांचिनी-संदीप द्वारा प्रतिपादित क्रांतिक घातांक ($2^* = \frac{2N}{2N-2}$) होता है [118]।

सोबोलेव क्रांतिक घातांक वाली समीकरणों के हलों के व्यवहार को समझना अत्यंत महत्वपूर्ण है क्योंकि ये समीकरण भौतिकी और अभियंत्रण के कई क्षेत्रों में प्रयुक्त होते हैं, जैसे कि नॉनलिनियर ऑप्टिक्स, बोस-आइंस्टीन संघनन, तरल यांत्रिकी आदि। इसके अतिरिक्त, इन समीकरणों का अध्ययन गणित में एक सक्रिय शोध क्षेत्र बना हुआ है जहाँ अब भी कई कठिन और खुले प्रश्न विद्यमान हैं। पी. एल. लायन्स द्वारा प्रतिपादित एकाग्रता-संपीडन [111,112] सिद्धांत के अनुसार, संपीडन की विफलता को तीन श्रेणियों में वर्गीकृत किया जाता है: एकाग्रता (जहाँ द्रव्यमान एक छोटे क्षेत्र में संकेंद्रित हो जाता है), लोप (जहाँ द्रव्यमान अनंत में विलीन हो जाता है), और द्वैधता (जहाँ द्रव्यमान दो अलग-अलग हिस्सों में विभाजित हो जाता है)। अतः, संपीडनीय डोमेन की तुलना में, यहाँ विश्लेषण के लिए अधिक उन्नत तकनीकों की आवश्यकता होती है, जैसे कि एकाग्रता-संपीडन सिद्धांत, माउंटेन पास प्रमेय, और पैले-स्माले शर्त [152]।

मांचिनी और संदीप के कार्य से प्रेरित होकर, जिसमें उन्होंने यह दिखाया कि पॉइंकारे-सोबोलेव असमानता के लिए हाइपरबोलिक स्थान में सम्बद्ध यूलेर-लाग्रांज समीकरण का एक अद्वितीय (हाइपरबोलिक समरूपता के सापेक्ष) सकारात्मक कमजोर हल होता है, हमने पहले उप-क्रांतिक अवस्था में इस समीकरण में व्यवधानों का अध्ययन किया। विशेष रूप से, हमने रैखिकता के पद में गैर-त्रिज्यात्मक व्यवधान के साथ संशोधित समीकरण का अध्ययन किया, दोनों सजातीय तथा असजातीय ($f \equiv 0, f \not\equiv 0$)परिप्रेक्ष्य में। इसके पश्चात, क्रांतिक घातांक की समस्याओं के अध्ययन में, हमने पहले क्रांतिक व्यवधान समस्या का विश्लेषण किया। उप-क्रांतिक स्थिति में, हाइपरबोलिक स्थानान्तरण [143] के कारण संपीडन की कमी देखी जाती है, जबकि क्रांतिक घातांक की समस्याओं में ऑबिन-तालेंती बबल्स की स्थानीय एकाग्रता के कारण भी संपीडन विफल हो सकता है।

इस समस्या के समाधान हेतु, हमने पैले-स्माले विघटन सिद्धांत की स्थापना की। इस वैश्विक संपीडन परिणाम का उपयोग करते हुए, हमने असजातीय पद वाले क्रांतिक घातांक मामलों $(f \not\equiv 0)$ में कई अस्तित्व और बहुलता परिणाम प्राप्त किए। इसके बाद, हमने शुद्ध रूप से क्रांतिक घातांक समस्या (i.e., $f \equiv 0$) का अध्ययन ए. बाहरी द्वारा प्रतिपादित अनंत पर स्थित क्रांतिक बिंदु की अवधारणा के माध्यम से किया। यह विश्लेषण अधिक जटिल था और इसके लिए कई सूक्ष्म अनुमानों के विकास की आवश्यकता थी।

अंततः, हाइपरबोलिक स्थान में हार्डी-लिटलवुड-सोबोलेव असमानता के परिप्रेक्ष्य में, हमने एक गैर-रेखिक चोकार्ड समीकरण के आधारभूत हलों की उपस्थिति और गुणात्मक विशेषताओं का अध्ययन किया।

Contents

List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{N}_0$	Set of non-negative integers
$\mathbb{R}$	Set of real numbers
$\mathbb{T}$	Torus
$\mathbb{C}$	Set of complex numbers
S^N	Cartesian product of N-copies of set S
$\mathbb{N}_0^N$	$\mathbb{N}_0^N = \mathbb{N}^N \cup \{0\}$
$\|u\|_X$	Norm of u in normed linear space X
$\|\cdot\|_r$	L^r-norm with respect to the volume measure for $1 \leq r \leq \infty$.
$A \Subset B$	A is a compact subset of B
$A \lesssim B$	If $\exists$ a constant $C > 0$ independent of appearing parameters such that $A \leq CB$
L^p	Lebesgue spaces, $1 \leq p \leq \infty$
H^s	Sobolev space of order s
C^α	Space of α-Hölder continuous function, $0 < \alpha < 1$
C^k	Space of k-times continuously differentiable functions
C^∞	Infinitely times differentiable functions
C_0^∞	Space of smooth functions with compact support