CONVERGENCE ANALYSIS OF REGULARIZED LEARNING ALGORITHMS FOR FUNCTIONAL DATA

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by

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Dedicated to
My Family

Certificate

This is to certify that the thesis entitled **"Convergence Analysis of Regularized Learning algorithms for Functional data"** submitted by **Mr. Naveen Gupta** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Date:
Place: New Delhi

Dr. S. Sivananthan
Professor
Department of Mathematics
Indian Institute of Technology Delhi

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"It always seems impossible until it's done." — *Nelson Mandela*

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New Delhi *Naveen Gupta*

Abstract

This thesis studies the convergence analysis of estimators in the Functional Linear Regression (FLR) and polynomial regression models, conducting a comprehensive examination of regularized learning algorithms within the framework of Functional Data Analysis (FDA). The study is motivated by the growing availability and importance of functional data in many scientific and applied fields, such as engineering, medicine, environmental studies, and finance, where observations are inherently represented as curves or functions across an arc. As the use of these data types increases, there is a growing need for learning algorithms that not only scale efficiently to high-dimensional functional inputs but also come with rigorous theoretical guarantees to ensure their reliability. The main contributions of the thesis are presented in Chapters 2 through 6.

In Chapter 2, we generalize classical regularization techniques—such as Tikhonov regularization—by utilizing a general spectral regularization method in the framework of reproducing kernel Hilbert spaces. We derive minimax optimal convergence rates for both estimation and prediction errors under commutative and non-commutative settings, significantly extending prior results by considering wider range of Hölder type smoothness conditions on the unknown slope function.

Chapters 3 and 4 address the computational challenges of large-scale kernel methods by exploring two computationally efficient algorithms in the reproducing kernel Hilbert space framework: the kernel Conjugate Gradient (CG) method and the Nyström subsampling method. The kernel CG method utilizes iterative optimization with early stopping rules to mitigate overfitting and reduce computational costs while maintaining optimal convergence rates. The Nyström method, on the other hand, applies low-rank kernel matrix approximations to significantly lower memory and runtime demands. We establish the optimal convergence rates for each algorithm, demonstrating their statistical efficiency alongside their computational benefits in respective chapters.

Chapter 5 extends the analysis of regularized learning algorithms to settings governed by general source conditions, moving beyond conventional assumptions such as operator monotonicity or Lipschitz continuity of the index functions. Moreover, the analysis is carried out in a general framework aimed at learning a vector in general separable Hilbert space.

Finally, Chapter 6 focuses on function-on-function polynomial regression models, which generalize the classical FLR framework by incorporating nonlinear interactions between functional predictors and responses. This extension enables the handling of more complex relationships that cannot be captured by linear models alone. In contrast to the kernel-based methods employed in earlier chapters, this chapter adopts a different framework by working directly in the $\mathbb{L}^2$ space. Theoretical analysis establishes convergence rates under general smoothness conditions. In addition, minimax lower bounds are derived, confirming the optimality of the proposed rates.

The contributions of this thesis collectively establish a thorough foundation for regularized functional regression. By addressing both statistical optimality and algorithmic efficiency, the work enhances the practical viability of functional learning methods while deepening their theoretical understanding.

सारांश

यह शोधप्रबंध **फलनात्मक रैखिक प्रतिगमन (Functional Linear Regression, FLR)** तथा **बहुपद प्रतिगमन (Polynomial Regression)** मॉडलों में अनुमानकों के **अभिसरण विश्लेषण** (convergence analysis) का अध्ययन करता है। इसमें **Functional Data Analysis (FDA)** के ढांचे के भीतर **नियमितीकृत लर्निंग एल्गोरिदम** का एक विस्तृत विश्लेषण प्रस्तुत किया गया है। यह अध्ययन विभिन्न वैज्ञानिक और अनुप्रयुक्त क्षेत्रों—जैसे कि **अभियांत्रिकी, चिकित्सा, पर्यावरणीय अध्ययन, तथा वित्त**—में फलनात्मक डेटा की बढ़ती उपलब्धता और प्रासंगिकता से प्रेरित है, जहाँ प्रेक्षण सामान्यतः वक्रों या फलनों के रूप में होते हैं।

जैसे-जैसे इन डेटा प्रकारों का उपयोग बढ़ता है, वैसे-वैसे ऐसे लर्निंग एल्गोरिदम की आवश्यकता महसूस की जा रही है जो **उच्च-आयामी फलनात्मक इनपुट** के लिए कुशलतापूर्वक कार्य कर सकें और जिनके लिए ठोस **सैद्धांतिक गारंटी** उपलब्ध हो। इस शोध के मुख्य योगदान **द्वितीय से षष्ठ अध्याय**तक प्रस्तुत किए गए हैं।

द्वितीय अध्याय में, पारंपरिक नियमितीकरण तकनीकों (जैसे **Tikhonov नियमितीकरण**) को सामान्यीकृत करते हुए **Reproducing Kernel Hilbert Spaces (RKHS)** के ढांचे में एक सामान्य **स्पेक्ट्रल नियमितीकरण विधि** का उपयोग किया गया है। हम संचालकों की अभिवर्तनीयता की स्थिति में और उसके बिना, दोनों ही स्थितियों में **अनुमान** और **पूर्वानुमान** की त्रुटियों के लिए **मिनीमैक्स इष्टतम अभिसरण दरें** प्राप्त करते हैं। यह कार्य **Hölder प्रकार की स्मूदनेस स्थितियों** की एक विस्तृत श्रेणी को शामिल करके पूर्व परिणामों का महत्वपूर्ण विस्तार करता है।

तृतीय और **चतुर्थ अध्याय** बड़े पैमाने पर केर्नल विधियों की संगणकीय चुनौतियों का समाधान प्रस्तुत करते हैं, जहाँ दो कुशल एल्गोरिदम—**केर्नल कॉन्जुगेट ग्रेडिएंट (CG)** विधि और **Nyström सबसैम्पलिंग** विधि—का विश्लेषण किया गया है। केर्नल CG विधि में **प्रारंभिक रोक नियम** (early stopping) के साथ पुनरावृत्त अनुकूलन (iterative optimization) का उपयोग किया गया है जिससे **overfitting** से बचा जा सके और संगणकीय लागत कम हो। Nyström विधि **low-rank kernel matrix approximation** द्वारा मेमोरी और रनटाइम को काफी कम करती है। दोनों ही विधियों के लिए **इष्टतम अभिसरण दरें** स्थापित की गई हैं।

पंचम अध्याय में हम **सामान्य स्रोत स्थितियों (general source conditions)** के अंतर्गत नियमितीकृत लर्निंग एल्गोरिदम का विश्लेषण करते हैं, जो पारंपरिक धारणाओं जैसे कि **operator monotonicity** या **index functions की Lipschitz continuity** से परे जाकर सामान्य **पृथक्करणीय हिल्बर्ट स्थानों** में कार्य करते हैं।

षष्ठ अध्याय में **फलन-पर-फलन बहुपद प्रतिगमन मॉडल (function-on-function polynomial regression)** का अध्ययन किया गया है, जो FLR फ्रेमवर्क का एक **गैर-रेखीय विस्तार** है। यह विस्तार अधिक जटिल संबंधों को संभालने में सक्षम है जिन्हें केवल रैखिक मॉडल से नहीं पकड़ा जा सकता। इस अध्याय में केर्नल आधारित विधियों की जगह सीधे **$\mathbb{L}^2$ स्पेस** में कार्य किया गया है।

यहां **सैद्धांतिक विश्लेषण** के द्वारा सामान्य स्मूदनेस स्थितियों में **अभिसरण दरें** प्राप्त की गई हैं, और साथ ही **मिनीमैक्स निम्न सीमाएँ** भी प्राप्त की गई हैं जिससे प्रस्तावित दरों की इष्टता की पुष्टि होती है।

इस शोधप्रबंध का समग्र उद्देश्य **फलनात्मक डेटा** के लिए **नियमितीकृत प्रतिगमन विधियों** की **सांख्यिकीय इष्टता** और **संगणकीय व्यावहारिकता** को एक साथ विकसित करना है, जिससे इन विधियों की सैद्धांतिक समझ को सुदृढ़ किया जा सके और उन्हें व्यवहार में लागू किया जा सके।

Contents

List of Figures

List of Tables

List of Symbols

Symbol	**Meaning**
$\mathbb{R}$	set of real numbers.
$\mathbb{N}$	set of natural numbers.
$\mathbb{R}^d$	d-dimensional Euclidean space.
β^*	true solution or the slope function.
$Y(\omega)$	output random variable
$X(\omega, \cdot)$	input stochastic process.
$L^2(S)$	space of square-integrable functions on S.
$\mathbb{E}$	Expectation.
$\langle \cdot, \cdot \rangle_{L^2}$	L^2 inner product.
ϵ	Noise or the error term.
φ	the continuous increasing index function.
f_ρ	regression function.
C	the covariance operator.
$\mathcal{H}$	reproducing kernel Hilbert space.
$k(\cdot, \cdot)$	reproducing kernel.
$k_x(\cdot)$	$k(x, \cdot)$.
J	inclusion operator from $\mathcal{H}$ to $L^2(S)$.
J^*	adjoint of the inclusion operator J.
$g_\lambda(\cdot)$	regularization family.
λ	regularization parameter.
$\hat{C}_n$	empirical covariance operator.
$\hat{\Lambda}_n$	$T^{\frac{1}{2}} \hat{C}_n T^{\frac{1}{2}}$.
$\hat{A}_n$	$J^* \hat{C}_n J$.
$\hat{R}$	$\frac{1}{n} \sum_{i=1}^n Y_i X_i$.

$\mathcal{R}(A)$	range space of an operator A.
$a_n \lesssim b_n$	$a_n \leq cb_n$ for some positive constant $c > 0$ and for all $n \in \mathbb{N}$.
$a_n \asymp b_n$	$a_n \lesssim b_n \lesssim a_n$.
$a \lesssim_p b$	$a \lesssim b$ with some probability law $\mathbb{P}$.
$\otimes$	Hilbert space tensor product.
$\oplus$	direct sum.
1_S	characteristic function of set S.
$\mathcal{K}(\cdot, \cdot)$	Kullback–Leibler (KL) divergence.
$\mathcal{K}_m$	order- m Krylov space.
$\\| \cdot \\|_{H_1 \to H_2}$	operator norm of an operator from a Hilbert space H_1 to Hilbert space H_2.
$\\| \cdot \\|_{HS}$	Hilbert Schmidt norm of an operator norm of an operator.

Abbreviation	**Meaning**
RKHS	reproducing kernel Hilbert space.
i.i.d.	independent identically distributed.
FPCA	functional principal component analysis.
FLR	functional linear regression.
FDA	fucntional data analysis.
r.h.s.	right hand side.
CG	Conjugate Gradient.