FINITE ELEMENT ANALYSIS OF NONLOCAL PARABOLIC AND HYPERBOLIC PROBLEMS OF KIRCHHOFF TYPE

VIMAL SRIVASTAVA

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

OCTOBER 2016

FINITE ELEMENT ANALYSIS OF NONLOCAL PARABOLIC AND HYPERBOLIC PROBLEMS OF KIRCHHOFF TYPE

by

VIMAL SRIVASTAVA

Department of Mathematics

Submitted

in fulfilment of the requirements of the degree of
Doctor of Philosophy

to the

Indian Institute of Technology Delhi
October 2016

Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Finite element analysis of nonlocal parabolic and hyperbolic problems of Kirchhoff type"** submitted by **"Mr. Vimal Srivastava"** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by him under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

Dr. Balaji Srinivasan
Associate Professor
Department of Applied Mechanics
IIT Delhi

Dr. V V K Srinivas Kumar
Assistant Professor
Department of Mathematics
IIT Delhi

New Delhi
October 2016

Acknowledgements

When I set out to pursue my Ph.D., I knew the work was going to be different than anything I had done before. But I never guessed how far I would travel beyond comfort of my home or how many people would guide me along the way. Reaching major milestones in one's life does not happen without the assistance of great many people, and for each one of them I am truly blessed and grateful.

I would like to express my sincere gratitude to my advisors Dr. V.V.K. Srinivas kumar and Dr. Balaji Srivivasan, for the continuous support of my Ph.D study and research, for their patience, motivation, enthusiasm, and immense knowledge. Their guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisors and mentors for my Ph.D study. I am privileged to have the opportunity to work under them.

I am thankful to IIT Delhi authorities for providing me the necessary facilities for smooth completion of my research work. I express my regards to Prof. R. K. Sharma, SRC Chairperson and former Head of the Department, for his support. Special thanks to my SRC (Student Research Committee) Members: Dr. B. Premchandran (Department of Mechanical Engineering) and Dr. Harish Kumar (Department of Mathematics) for generously sharing their time and knowledge. I'm also very much thankful to Prof. K. Höllig (Fachbereich Mathematik, IMNG Universität Stuttgart), Dr. T. Gudi (Department of Mathematics IISc Bangalore) and Dr. K. Sreenadh (Department of Mathematics IIT Delhi) for their help and suggestions during my research work. I'm grateful to all faculty members of Department of Mathematics, IIT Delhi, for their co-operation and support.

Words cannot express how grateful I am to my daughters Keshavi and Kashika, my mother, Late *father, mother-in law, brothers and sisters-in-law for all of the sacrifices that they have made on my behalf.*

I would also like to thank all of my friends who supported me in writing, and incented me to strive towards my goal.

A special thanks to Dr. V.K.Srivastava Sir (Former Dean IR&D, IIT Delhi), who encouraged me to pursue research. I am also thankful to my research collaborator Dr. sudhakar without his help this work was very difficult. I would like to express my special thanks to Mr.Ashu Goel (Management Trustee), Dr. Vipin Garg (Advisor), Dr. Bhavesh Kumar Chauhan (Director) ABES Institute of Technology, who gave me opportunity by providing me necessary facility to pursue my Ph.D. work.

At the end I would like express appreciation to my beloved wife Nimisha who spent sleepless nights with and was always my support in the moments when there was no one to answer my queries.

Finally, I thank the almighty God for the passion, strength, perseverance, and the resources to complete this study.

New Delhi *Vimal Srivastava*

October 2016

Abstract

Partial differential equations (PDEs) have many important applications in various fields of science and engineering, such as fluid dynamics, materials science, inverse problems, computer graphics, and pattern formation. Partial differential equations are described as relation between the value of an unknown function and its derivatives of different order. For the partial differential equation to hold at a point, it needs value of function in an arbitrarily small neighborhood, so that all the derivatives of desired order can be computed. On the other hand, for a nonlocal partial differential equation to hold at a point, information about the value of function far from that point is needed. In general it happens because such equations involve an integral operator.

The word nonlocal has become a keyword in the area of nonlinear analysis. Due to wide range of applications in the real world, nonlocal problems are being very extensively studied since last decade [9–11, 14, 26, 36, 43].

Analytical solution of these type of problems is either not known or very difficult to develop. Therefore, one should look for accurate and efficient numerical methods to solve them. There is enough literature available on well-posedness of nonlocal problems at continuous level. However, few studies towards numerical approximations are made, and these are restricted to elliptic [21] and parabolic problems [1, 37]. In this thesis work our aim is to find numerical approximation of nonlocal parabolic and hyperbolic problems. Finite element method is one of the numerical techniques which may be used to solve the steady state as well as time dependent problems. In finite element method, the discrete weak formulation of nonlocal problems (under consideration) leads to a system of nonlinear algebraic equations. We can solve this system either by Picard's iterative method or by Newton's method. Newton's method has an advantage over Picard's method in the sense of rate of convergence. Newton's method is faster and therefore we can get required accuracy in less number of iterations. An important issue while using Newton's method in the numerical simulation of nonlocal problems is related to its structure. In fact, unlike in the case of local problems where the Jacobian matrix is sparse and banded, in nonlocal case Jacobian matrix is dense. In other words, nonlocal term in nonlocal problems destroys the sparsity of Jacobian matrix. To overcome this difficulty, the author in [21] gave a new methodology. In this methodology author suggest that sparse Jacobian matrix can be obtain by rewriting equation in different way and authour confirm this by taking elliptic nonlocal problem of Kirchhoff type.

Motivated by this we successfully extended this technique for time dependent problems. We first consider a nonlocal parabolic equation. For time discretization backward Euler as well as Crank-Nicolson schemes are used. We obtain sparse Jacobian matrix after applying Newton's method to the fully discrete formulation of the given parabolic nonlocal problem. We provide the existence and uniqueness results at the continuous as well as at the discrete levels. We prove *a priori* error bounds for semidiscrete and fullydiscrete formulation and verify these results by numerical simulation.

We also consider a nonlocal parabolic coupled system of equations. We discuss the existence and uniqueness results at the continuous as well as discrete levels and we give *a priori* error bounds at semidiscrete and fullydiscrete formulations. Again we apply Newton's methods to deal with nonlocal terms by extending the given technique in [21] to get sparse Jacobian matrix for elliptic problemc. Theoretical

estimates are confirmed via numerical experiments.

Further we obtain finite element solution for a nonlocal hyperbolic problem of Kirchhoff type. For time discretization backward Euler scheme is used. Here also we get sparse Jacobian matrix after using Newton's method in the reformulated equations. We discuss existence and uniqueness results at continuous as well as at the discrete levels [34]. Numerically we have obtained convergence in $H^1(\Omega)$ norm.

Contents

List of Figures

List of Tables

List of Symbols

Symbol	**Meaning**
$\mathbb{N}$	Set of *natural numbers*
$\mathbb{R}$	Set of *real numbers*
$\forall x$	For *all* x
$x \in X$	x is a *member* of X
Δ	Laplacian
∇	Gradient
M'	Dual Space of M
$\mathbf{B}^T$	Transpose of $\mathbf{B}$
$H_0^k(\Omega)$	Trace zero elements of Sobelev Space $H^k(\Omega)$
$L_0^2(\Omega)$	Mean zero elements of $L^2(\Omega)$
$\rightharpoonup$	Weakly convergent
$\overset{*}{\rightharpoonup}$	Star weakly convergent
$\|\|w\|\|_m$	$\left(\sum_{0\leq\alpha\leq m} \|\|\frac{\partial^\alpha w}{dx^\alpha}\|\|^2\right)^{1/2}$