A THESIS ON

NUMERICAL EVALUATION OF CAUCHY TYPE SINGULAR INTEGRA

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C E R T I F I C A T E

This is to certify that the thesis entitled "Numerical Evaluation of Cauchy Type Singular Integrals" which is being submitted by Mr. Ramakrishnan T.R. for the award of Doctor of Philosophy in Mathematics to the Indian Institute of Technology, New Delhi, is a record of bonafide research work. He has worked for the last three years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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S Y N O P S I S

Singular integrals of Cauchy type occur often in the study of aerodynamics. While the theory of quadrature for functions of one variable is fully developed, so far no systematic method has been presented for the development of quadrature formulas for Cauchy principal value integrals. Most of the previous attempts have been to use special techniques for obtaining quadrature formulas for the Cauchy principal value integral $\int_{-1}^{1} (x-c)^{-1} f(x)dx$, $c \in (-1,1)$. In this thesis we discuss two methods for the development of quadrature formulas for Cauchy principal value integrals; the first method is to make use of the Cauchy residue theorem while the second method makes use of the idea of subtracting out the singularity. In Chapter I, using the complex variable methods we obtain modified Gauss-Jacobi quadrature formulas for the numerical evaluation of $\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)dx$, $\alpha, \beta > -1$ where $f(x)$ has a number of simple poles in $(-1,1)$. In Chapter II, using the complex variable methods, we modify the trapezoidal sequence of rules for the numerical evaluation of integrals of periodic functions with Cauchy and Poisson type kernels. The errors of modified Gauss-Jacobi quadrature formulas and the modified trapezoidal rules are also obtained; the contour integral expressions for these errors are similar to those for the errors of the Gauss-Jacobi and the trapezoidal rules. Thus, in Chapter III, we include asymptotic estimates for the errors of Gauss-Jacobi quadrature formulas and the trapezoidal sequence of rules for certain types of analytic functions. In Chapter IV, we give a systematic method, based on the idea of subtracting out the singularity, for the construction of quadrature formulas for Cauchy principal value integrals. We show that, using the method of subtracting out the singularity, a known quadrature formula for the proper integral $\int_a^b w(x)f(x)dx$ can be easily modified to obtain quadrature formulas for the Cauchy principal value integral $\int_a^b w(x)(x-c)^{-1}f(x)\,dx$, $-\infty \leq a < c < b \leq \infty$. Various particular modified formulas are discussed for Cauchy principal value integrals with finite, semi-infinite and infinite range of integration. Most of the results previously obtained by various authors are included as particular cases of our results. Finally we show that, using the method of subtracting out the singularity, the trapezoidal sequence of rules is very easily modified for the numerical evaluation of $\int_0^{2\pi} f(x)\cot(\frac{x-c}{2})\,dx$, $c \in [0, 2\pi)$, $f(x)$ periodic with period 2π. Numerical examples are included to illustrate the modified quadrature formulas obtained.

The thesis consists of 4 chapters and a brief description of the contents of each chapter follows.

Chapter I: Using the complex variable methods we obtain modified Gauss-Jacobi quadrature formulas for the Cauchy principal value integral $\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}f(x)dx$, $\alpha,\beta > -1$, where $f(z)$ is analytic in some region containing the interval $[-1,1]$ except for a finite number of simple poles, one or more of these lying in $(-1,1)$. Modified quadrature formulas are developed corresponding to the cases when none of the poles coincides with a Jacobi abscissa and when one or more poles of $f(z)$ coincide with a Jacobi abscissa. In each case the modified formula consists of Gauss-Jacobi quadrature formula applied to $f(x)$ and certain 'correction' terms which involve 'functions of the second kind'. If m denotes the number of poles of $f(z)$, then an n-point modified Gauss-Jacobi formula has polynomial precision 2n+m-1. Some explicitly determined formulas are listed for $(\alpha,\beta) = (\pm\frac{1}{2}, \pm\frac{1}{2})$ and some interesting side results are derived. Contour integral representations of the errors of the modified Gauss-Jacobi quadrature formulas are given.

Chapter II: Using the complex variable methods we modify the trapezoidal sequence of rules for the numerical evaluation of the Cauchy principal value integral $\int_{0}^{2\pi} f(x) \times \cot(\frac{x-c}{2})dx$, $c \in [0, 2\pi)$, $f(x)$ periodic with period 2π. For $c \neq \frac{k\pi}{n}$, $k = 0(1)2n-1$, a 2n-point modified trapezoidal formula consists of the trapezoidal rule applied to $f(x)\cot(\frac{x-c}{2})$ and the 'correction' term $2\pi f(c)\cot nc$.

Modified trapezoidal rules are also obtained for the case when $c = \frac{k\pi}{n}$ for some $k = 0(1)2n-1$. We then obtain modified trapezoidal sequence of rules for the numerical evaluation of integrals of periodic functions with Poisson type kernel $\int_0^{2\pi} (\cosh b - \cos(x-c))^{-1} f(x)\,dx$, $b \neq 0$. The trapezoidal sequence of rules gives good approximations to the above integral for large values of b and these approximations get worse for b getting closer to zero. However, the modified trapezoidal sequence of rules obtained gives much better approximate values for the integral even for b close to zero. Contour integral representation for the errors of the modified trapezoidal sequence of rules are given in each case.

Chapter III: The errors of modified Gauss-Jacobi quadrature formulas of Chapter I and the errors of the modified trapezoidal sequence of rules of Chapter II are similar to the errors of the corresponding 'unmodified' quadrature formulas. Therefore, inthis chapter we discuss obtaining asymptotic estimates for the errors of Gauss-Jacobi and the trapezoidal rule for certain types of analytic functions.

Chapter IV: In this chapter a systematic development of quadrature formulas for Cauchy principal value integrals is given using the method of subtracting out the singularity. We show that a known quadrature formula for the proper integral

$\int_a^b w(x)f(x)dx$ can be easily modified for the numerical evaluation of the Cauchy principal value integral $⨍_a^b w(x)(x-c)^{-1}f(x)dx$, $-\infty \leqslant a < c < b \leqslant \infty$. The computational usefulness of such a modified quadrature formula depends upon the availability of $q_0(c) = ⨍_a^b w(x)(x-c)^{-1}dx$; fortunately, for some of the more commonly used (classical) weight functions $w(x)$, $q_0(c)$ can be obtained explicitly. We obtain several particular modified quadrature formulas for Cauchy principal value integrals with finite, semi-infinite and infinite range of integration. Most of the previously obtained formulas by various authors, mainly for integrals with finite range of integration, are included here as particular cases of our modified formulas. Extensions to the case of a number of poles is included. We then show that using the method of subtracting out the singularity, the trapezoidal sequence of rules for the proper integral $\int_0^{2\pi} f(x)dx$, $f(x)$ periodic with period 2π can be easily modified to obtain quadrature formulas for $⨍_0^{2\pi} f(x)\cot(\frac{x-c}{2})dx$, $c \in [0, 2\pi)$.

Some of the results of the thesis have been published in BIT (1974), Numerische Mathematik (1974) and in Calcolo (1973).

C O N T E N T S

Note : An equation numbered (n) occurring in Chapter m will be referred to as equation (m.n); similarly for theorems , remarks etc.