NUMERICAL METHODS FOR MULTIPLE SOLUTIONS OF SOME p-LAPLACIAN PROBLEMS

SUCHISMITA PATRA

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

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by

SUCHISMITA PATRA

DEPARTMENT OF MATHEMATICS

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"Numerical Methods for Multiple Solutions of Some *p*-Laplacian Problems"** submitted by **Ms. Suchismita Patra** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

V.V.K. Srinivas Kumar

Dr. V. V. K. Srinivas Kumar
Assistant Professor
Department of Mathematics
Indian Institute of Technology Delhi

New Delhi
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Suchismita Patra

Suchismita Patra

Abstract

The widespread applications of the p-Laplacian problems with multiple solutions in real life naturally appeals to the development of numerical methods for their solutions. Many significant solutions to these problems are known to lack stability. As a consequence, they are very elusive to capture and also hard to study. Most often, to solve such problems numerically for multiple solutions, there are two things to keep in mind. Firstly, we need an algorithm that can capture multiple unstable solutions in a stable way. Secondly, implementing those algorithms through computers requires a different set of numerical techniques for discretization of the problems. Although the p-Laplacian problems with multiple solutions refer to a large class of problems, here we consider only a few problems of that class for the development of appropriate numerical methods to obtain their multiple solutions. Precisely, in this thesis we study the numerical methods for finding multiple solutions to four types of p-Laplacian problems: p-Laplacian equations, cooperative p-Laplacian systems and the eigenvalue problems for the p-Laplacian and for a system of p-Laplacians.

By considering p-area problems, where the p-Laplacian equation is a special case of such problems, a numerical algorithm which is a hybridization of the local minimax method and Newton's method is developed for finding their multiple solutions. We discuss a mathematical justification for the applicability of the method which highlights the main advantages of using this hybrid method compared to the existing well-known local minimax method. The discretization of the problems using the finite element method is presented with the associated convergence analysis of the solutions to the discretized problems. Thereafter, two numerical methods are developed for the system of p-Laplacians by elevating the strategy of the min-orthogonal method. One is for the cooperative p-Laplacian systems and the other one is for the eigenvalue problems for a system of p-Laplacians. In both cases, we first recommend a selection mapping in a product Banach space so that the solution manifold is well-defined. We then set up local characterizations for the coexisting solutions of p-Laplacian systems and the eigenfunctions of the p-Laplacian system, respectively. These local characterizations prompts us to designed two algorithms which are elucidated by their flow charts. Further, we carry out the convergence analysis of these algorithms. We also develop some techniques for local instability analysis of the solutions to both types of problems that can be done computationally. Then we discuss the finite element discretization of both the problems and give the convergence results of their respective numerical solutions. In numerical experiments, all of these numerical methods have been successfully applied to solve different types of model problems with multiple solutions.

For the culmination of this piece of work we carry out the numerical implementation of a mesh-free strategy called WEB-method for obtaining multiple solutions of some p-Laplacian problems. Basically, to avoid the drawbacks of the standard mesh-based finite element method we applied the WEB-method for the discretization of problems. The WEB-method is a mesh-less Galerkin method that uses weighted extended B-splines on regular grids as basis functions. The efficiency of the WEB-method is tested by combining it with a local minimax algorithm to solve two different model problems, namely the p-Laplacian equations and the eigenvalue problem for the p-Laplacian for their multiple numerical solutions. The experimental results illustrate advantages of using the WEB-method over the standard mesh-based finite element method though sometimes WEB-method is computationally expensive.

सार

वास्तविक जीवन में कई समाधानों के साथ पी–लाप्लासियन समस्याओं के व्यापक अनुप्रयोग स्वाभाविक रूप से उनके समाधान के लिए संख्यात्मक तरीकों के विकास की अपील करते हैं। इन समस्याओं के कई महत्वपूर्ण समाधान स्थिरता की कमी के लिए जाने जाते हैं। नतीजतन, वे गणना करने के लिए बहुत मुश्किल हैं और अध्ययन करने में भी कठिन हैं। बहुधा, ऐसी समस्याओं को संख्यात्मक रूप से कई समाधानों के लिए हल करने के समय, दो बातों को ध्यान में रखना चाहिए। सबसे पहले, हमें एक एल्गोरिथ्म की आवश्यकता है जो स्थिर तरीके से कई अस्थिर समाधानों को पकड़ सके। दूसरे, कंप्यूटर के माध्यम से उन एल्गोरिदम को लागू करने के लिए समस्याओं के पृथक्करण के लिए संख्यात्मक तकनीकों के एक अलग सेट की आवश्यकता होती है। हालाँकि कई समाधानों के साथ पी–लाप्लासियन समस्याएं समस्याओं के एक बड़े वर्ग को संदर्भित करती हैं, यहाँ हम उनके कई समाधानों को प्राप्त करनेका उपयुक्त संख्यात्मक विधियों के विकास के लिए उस वर्ग की केवल कुछ समस्याओं पर विचार करते हैं। विशेष रूप से, इस थीसिस में हम चार प्रकार की पी–लाप्लासियन समस्याओं के कई समाधान खोजने के लिए संख्यात्मक विधियों का अध्ययन करते हैं: पी–लाप्लासियन समीकरण, कोआपरेटिव पी–लाप्लासियन सिस्टम और पी–लाप्लासियन के लिए और पी–लाप्लासियन की एक सिस्टम के लिए आइगनवेल्यू समस्याएं।

पी–एरिया की समस्याओं पर विचार करके, जहां पी–लाप्लासियन समीकरण ऐसी समस्याओं का एक विशेष प्रकार है, एक संख्यात्मक एल्गोरिथ्म जो लोकल मिनिमैक्स विधि और न्यूटन की विधि का एक संकरण है उनके कई समाधान खोजने के लिए विकसित की गई है। हम उस विधि की प्रयोज्यता के लिए गणितीय औचित्य पर चर्चा करते हैं जो मौजूदा प्रसिद्ध लोकल मिनिमैक्स विधि की तुलना में इस संकर विधि का उपयोग करने के मुख्य लाभों पर प्रकाश डालता है। फहैनाईट एलिमेंट विधि का उपयोग करके समस्याओं का पृथक्करण पृथकृत समस्याओं के समाधान के संबद्ध अभिसरण विश्लेषण के साथ प्रस्तुत किया जाता है। इसके बाद, मिन–ऑर्थोगोनल विधि की कार्यनीति को बढ़ाकर पी–लाप्लासियन सिस्टम के लिए दो संख्यात्मक तरीके विकसित किए जाते हैं। एक कोआपरेटिव पी–लाप्लासियन सिस्टम के लिए है और दूसरा पी–लाप्लासियन सिस्टम की आइगनवेल्यू समस्याओं के लिए है। दोनों ही मामलों में, हम पहले एक उत्पाद बनच स्पेस में सिलेक्शन मैपिंग की सलाह देते हैं ताकि सलूशन मनिफॉल्ड अच्छी तरह से परिभाषित हो। इसके बाद हमने क्रमानुसार में पी–लाप्लासियन सिस्टम के कोएक्सिस्टिंग समाधान और पी–लाप्लासियन सिस्टम के आइगनफैक्शन के लिए स्थानीय लक्षण वर्णन स्थापित किए। ये स्थानीय लक्षण हमें दो एल्गोरिदम डिज़ाइन करने के लिए प्रेरित करते हैं जो उनके फ़्लो चार्ट द्वारा स्पष्ट किया गया हैं। इसके अलावा, हम इन एल्गोरिदम के अभिसरण विश्लेषण को अंजाम देते हैं। हम दोनों प्रकार की समस्याओं के समाधान के स्थानीय अस्थिरता विश्लेषण के लिए कुछ तकनीकों का विकास भी करते हैं जो कम्प्यूटेशनल रूप से किया जा सकता है। फिर हम दोनों समस्याओं के फहैनाईट एलिमेंट पृथक्करण पर चर्चा करते हैं और उनके संबंधित संख्यात्मक समाधानों के अभिसरण परिणाम प्रदान करते हैं। संख्यात्मक प्रयोगों में, इन सभी संख्यात्मक विधियों को सफलतापूर्वक कई समाधानों के साथ विभिन्न प्रकार की मॉडल समस्याओं को हल करने के लिए लागू किया गया है।

काम के इस टुकड़े की परिणिति के लिए हम कुछ पी–लाप्लासियन समस्याओं के कई समाधान प्राप्त करने के लिए वेब–विधि नामक एक जाल–मुक्त कार्यनीति का संख्यात्मक कार्यान्वयन करते हैं। मूल रूप से, मानक जाल–आधारित फहैनाई एलिमेंट विधि की कमियों से बचने के लिए हमने समस्याओं के पृथक्करण के लिए वेब–विधि लागू की। वेब–विधि एक जाल–मुक्त गैलेरकिन विधि है जो बेसिस फंक्शन के रूप में नियमित ग्रिड पर वेटेड एक्सटेंडेड बी–स्प्लीन का उपयोग करती है। वेब–विधि की दक्षता को दो अलग–अलग मॉडल समस्याओं को हल करने के लिए अर्थात् पी–लाप्लासियन समीकरण और पी–लाप्लासियन आइगनवेल्यू समस्या उनके कई संख्यात्मक समाधानों के लिए, एक लोकल मिनिमैक्स एल्गोरिथ्म के साथ जोड़कर परीक्षण किया जाता है। प्रयोगात्मक परिणाम मानक जाल–आधारित फहैनाई एलिमेंट विधि पर वेब–विधि का उपयोग करने के फायदे बताते हैं, हालांकि कभी–कभी वेब–विधि कम्प्यूटेशनल रूप से महंगी होती है।

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Symbol	Meaning
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\mathbb{N}_0$	Set of non-negative integers
$\forall x$	For all x
$x \in X$	x is a member of X
$\preceq, \succeq, \asymp$	Inequalities upto constants
2^X	Power set of X
B^*	Dual space of B
∇	Gradient
Δ	Laplacian
Δ_p	p-Laplacian
$\Omega \subset \mathbb{R}^\ell$	Bounded domain in $\mathbb{R}^\ell$
$\bar{\Omega}$	Closure of Ω
$\partial\Omega$	Boundary of the domain Ω
$dist$	Distance functions
$supp$	Support of a function
C^ℓ	Space of ℓ-times continuously differentiable functions
$C_c^\infty(\Omega)$	Space of functions in $C^\infty(\Omega)$ with compact support in Ω
$L^2(\Omega)$	Space of square integrable functions
$W^{m,p}(\Omega)$	Sobelev space of order m for $1 < p < \infty$
$W_0^{m,p}(\Omega)$	Trace zero elements of Sobelev space $W^{m,p}(\Omega)$
$H^m(\Omega)$	Sobelev space $W^{m,2}(\Omega)$
$H_0^m(\Omega)$	Trace zero elements of Sobelev space $H^m(\Omega)$