A THESIS ON

ESTIMATION OF ERRORS OF NUMERICAL INTEGRATION FOR ANALYTIC FUNCTIONS

By

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C E R T I F I C A T E

This is to certify that the thesis entitled 'Estimation of Errors of Numerical Integration for Analytic Functions' which is being submitted by Mr. Man Mohan Chawla for the award of Degree of Doctor of Philosophy (Mathematics) to the Indian Institute of Technology, Delhi, is a record of bonafide research work. He has worked for the last three years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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S Y N O P S I S

Classical error estimates for the rules of approximate integration using derivatives can be used but they are not of great practical value since the derivatives are not usually available. In this thesis we have studied the analytic function theory methods for the estimation of errors of numerical integration formulas - Gauss type quadratures, Gaussian cubature formulas and certain Lagrangian quadratures (e.g., the Clenshaw-Curtis quadrature). Davis' method for the estimation of quadrature errors based on Hilbert space techniques has also been analysed. In contrast to the usual real-variable theory methods, these methods do not involve the use of the higher derivatives of the function but require instead a knowledge of the size of the integrand in the complex plane. These estimates will therefore be of practical value.

A description of the contents of six chapters included in the thesis follows.

CHAPTER I: We have established simple derivative-free error estimates for the Gauss-Legendre, the two Gauss-Chebyshev and a Gauss-Jacobi quadrature applied to analytic functions. A few lemmas giving asymptotic expansions, which are useful for the derivation of these estimates, have also been proved. The estimates obtained improve upon certain known error estimates for these Gauss type quadratures. Contour integral representations obtained for the errors $E_n(f)$ have the advantage that, in specific cases, it may be possible, by a further analysis on the evaluation of the contour integral, to obtain $E_n(f)$, possibly exactly or asymptotically (Chapter II), depending upon the nature of the function in the complex plane; as, for example, in the case of functions with poles. This is particularly true of the Chebyshev error formulas which are very simply expressed with contours as certain ellipses.

CHAPTER II: We have studied the asymtotics of the Gaussian quadrature error obtaining estimates according to the nature of the integrand in the complex plane: entire functions, functions with poles, function having singularities on the real axis - branch point, singularity at an end-point of the interval of integration, or a logarithmic singularity. The analysis also brings out the effect of the nature of $f(z)$ on the rate of convergence of the Gauss quadrature formula.

CHAPTER III: We continue the study of Chapter I to discuss the estimation of errors and convergence of Gaussian quadrature formulas of the closed type - Lobatto, Radau and a Gauss-Chebyshev quadrature formula of the closed type. A few lemmas required to obtain these estimates have also been established.

CHAPTER IV: Error estimates have been obtained for the two Lagrangian quadrature schemes, applied to analytic functions, based respectively on the "classical and "practical" abscissas.

CHAPTER V: Error estimates through Davis' method employing the "double integral" norm as well as the "line integral" norm are obtained for Gauss type quadratures whose abscissas and weights are simly expressed. Error-functional norms for these quadratures, as also for the trapezoidal and Simpson rules, have been 'explicitely' evaluated. The line integral norm error estimates obtained through Davis' method are essentially the same as those obtained through the analytic function theory in Chapter I.

CHAPTER VI: Two-term contour integral expressions are obtained for the error of Gaussian cubature formulas, which represent generalization of the known quadrature error formulas and improves upon the known cubature error formula. Derivative-free error estimates are obtained for the Gauss-Legendre and Gauss-Chebyshev cubature formulas.

The thesis is based on the following eight papers.

1. "Error Estimates for Gauss Quadrature Formulas for Analytic Functions," Mathematics of Computation, January 1968.

2. "Asymptotic Error Estimates for the Gauss Quadrature Formula," Mathematics of Computation, January 1968.

3. "Error Estimates for the Clenshaw-Curtis Quadrature," Mathematics of Computation, July 1968.

4. "Error Bounds for the Gauss-Chebyshev Quadrature Formula of the Closed Type," Mathematics of Computation, July 1968.

5. "On Davis' Method for the Estimation of Errors of Gauss Type Quadratures," Mathematics of Computation communicated by Professor Philip J. Davis.

6. "On the Chebyshev Polynomials of the Second Kind," SIAM Review, October 1967.

7. "On the Estimation of Errors of Gaussian Cubature Formulas," SIAM Journal on Numerical Analysis, March 1968.

8. "A note on the estimation of the coefficients in the Chebyshev series expansion of a function having a logarithmic singularity," Computer Journal, Vol. 9, 1967, p. 413.

C O N T E N T S

Note. A reference to Th. n of Chapter m is made as Th. $m.n$, same for Lemmas, etc. Similarly, Eqn. $(m.n)$ means Eqn. numbered n of Chapter m.