FINITE ELEMENT METHODS FOR LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS GOVERNED BY ELLIPTIC OPERATORS

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DEPARTMENT OF MATHEMATICS
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FINITE ELEMENT METHODS FOR LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS GOVERNED BY ELLIPTIC OPERATORS

by

Divay

Department of Mathematics

Submitted

In fulfilment of the requirements of the degree of
Doctor of Philosophy
to the

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Dedicated to
my mother

Certificate

This is to certify that the thesis entitled **FINITE ELEMENT METHODS FOR LINEAR QUADRATIC OPTIMAL CONTROL PROBLEMS GOVERNED BY ELLIPTIC OPERATORS** submitted by **Mr. Divay** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
January 2024

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Abstract

The primary focus of this thesis is to develop the error analysis of finite element methods (FEMs) for linear quadratic optimal control problems governed by elliptic partial differential equations. In this context, our attention is directed towards two control problems namely distributed control problem and Dirichlet boundary control problem. The main objective of this research is to investigate the accuracy and convergence properties of FEMs applied to optimal control problems. We analyze the errors introduced by the discretization process and provide insights into their behaviour as the mesh is refined. A rigorous theoretical framework is developed to establish error estimates and convergence rates for the state, adjoint state and control variables. These estimates quantify the accuracy of the numerical approximation and provide crucial information for assessing the reliability of the computed solutions. Furthermore, the thesis addresses the issue of adaptivity in FEMs for optimal control problems. Adaptive techniques play a crucial role in balancing computational cost and accuracy by refining the mesh selectively in regions of interest. Theoretical results and numerical experiments presented in this thesis highlight the capabilities and limitations of FEMs for solving optimal control problems. The findings of this research contribute to the understanding of FEMs for optimal control problems and provide valuable guidance for practitioners in selecting appropriate numerical methods and designing efficient computations.

This thesis comprises seven chapters, encompassing an introductory chapter and a concluding chapter. Chapter 1 is dedicated to reviewing relevant literature, gathering established results, and introducing essential preliminaries that serve as fundamental components for the subsequent chapters. In Chapter 2, we study HCT and Morley finite element methods for the distributed elliptic optimal control problem with point-wise state constraints. In this chapter, we carry out *a posteriori* error analysis in the energy norm and discuss the reliability and efficiency of the proposed *a posteriori* error estimator. In Chapter 3, we analyze mixed finite element method to derive *a priori* and *a posteriori* error estimates for a second order Dirichlet boundary control problem. The analysis primarily relies on the use of several auxiliary problems and Helmholtz decomposition. Chapter 4 presents the derivation of *a priori* and *a posteriori* error estimates for a specific class of discontinuous Galerkin (DG) methods applied to the second order Dirichlet boundary control problem. A crucial step in the analysis involves the meticulous selection of a discrete cost functional that appropriately approximates the continuous cost functional, leveraging the symmetric property of the bilinear form. This careful selection ensures the accuracy and effectiveness of the analysis. In Chapter 5, we delve into the examination of the control constrained Dirichlet boundary control problem, which is governed by a second-order elliptic partial differential equation (PDE). This problem stands in contrast to the one discussed in Chapter 3 and Chapter 4, where the governing equation remains a second-order elliptic PDE, but there are no constraints imposed on the control variable. The symmetric class of DG methods is analyzed to derive both *a priori* and *a posteriori* error estimates. In Chapter 6, symmetric interior penalty Galerkin (SIPG) finite element method is analyzed to carry out *a priori* and *a posteriori* error analysis for fourth order Dirichlet boundary control problem. When the penalty parameters are chosen suitably, the complete fourth order SIPG finite element method reduces to the classic C^0-interior penalty method also. The corresponding error estimates (both *a priori* and *a posteriori*) are obtained for C^0-interior penalty method. At the end of each chapter from Chapter 2 to Chapter 6, numerical results are showcased to provide practical illustrations of the theoretical findings established in their respective chapters. In Chapter 7, we provide a summary of the completed work and suggest potential extensions and propose future directions for the work presented.

सार

इस थीसिस का प्राथमिक फोकस अण्डाकार आंशिक अंतर समीकरणों द्वारा शासित रैखिक द्विघात इष्टतम नियंत्रण समस्याओं के लिए परिमित तत्व विधियों (एफईएम) के त्रुटि विश्लेषण को विकसित करना है। इस संदर्भ में, हमारा ध्यान दो नियंत्रण समस्याओं की ओर केंद्रित है, अर्थात् वितरित नियंत्रण समस्या और डिरिचलेट सीमा नियंत्रण समस्या। इस शोध का मुख्य उद्देश्य इष्टतम नियंत्रण समस्याओं पर लागू एफईएम की सटीकता और अभिसरण गुणों की जांच करना है। हम विवेकाधीन प्रक्रिया द्वारा शुरू की गई त्रुटियों का विश्लेषण करते हैं और जाल को परिष्कृत करते समय उनके व्यवहार में अंतर्दृष्टि प्रदान करते हैं। राज्य, आसन्न राज्य और नियंत्रण चर के लिए त्रुटि अनुमान और अभिसरण दर स्थापित करने के लिए एक कठोर सैद्धांतिक ढांचा विकसित किया गया है। ये अनुमान संख्यात्मक अनुमान की सटीकता की मात्रा निर्धारित करते हैं और गणना किए गए समाधानों की विश्वसनीयता का आकलन करने के लिए महत्वपूर्ण जानकारी प्रदान करते हैं। इसके अलावा, थीसिस इष्टतम नियंत्रण समस्याओं के लिए एफईएम में अनुकूलनशीलता के मुद्दे को संबोधित करती है। अनुकूली तकनीकें रुचि के क्षेत्रों में चयनात्मक रूप से जाल को परिष्कृत करके कम्प्यूटेशनल लागत और सटीकता को संतुलित करने में महत्वपूर्ण भूमिका निभाती हैं। इस थीसिस में प्रस्तुत सैद्धांतिक परिणाम और संख्यात्मक प्रयोग इष्टतम नियंत्रण समस्याओं को हल करने के लिए एफईएम की क्षमताओं और सीमाओं पर प्रकाश डालते हैं। इस शोध के निष्कर्ष इष्टतम नियंत्रण समस्याओं के लिए एफईएम की समझ में योगदान करते हैं और उपयुक्त संख्यात्मक तरीकों का चयन करने और कुशल गणनाओं को डिजाइन करने में चिकित्सकों के लिए मूल्यवान मार्गदर्शन प्रदान करते हैं।

इस थीसिस में सात अध्याय हैं, जिसमें एक परिचयात्मक अध्याय और एक समापन अध्याय शामिल है। अध्याय 1 प्रासंगिक साहित्य की समीक्षा करने, स्थापित परिणाम

एकत्र करने और आवश्यक प्रारंभिकताओं को प्रस्तुत करने के लिए समर्पित है जो बाद के अध्यायों के लिए मौलिक घटकों के रूप में काम करते हैं। अध्याय 2 में, हम बिंदु–वार राज्य बाधाओं के साथ वितरित अण्डाकार इष्टतम नियंत्रण समस्या के लिए एचसीटी और मॉर्ले परिमित तत्व विधियों का अध्ययन करते हैं। इस अध्याय में, हम ऊर्जा मानदंड में एक पश्चवर्ती त्रुटि विश्लेषण करते हैं और प्रस्तावित पश्चवर्ती त्रुटि अनुमानक की विश्वसनीयता और दक्षता पर चर्चा करते हैं। अध्याय 3 में, हम दूसरे क्रम डिरिच–लेट सीमा नियंत्रण समस्या के लिए एक प्राथमिकता और एक पश्चवर्ती त्रुटि अनुमान प्राप्त करने के लिए मिश्रित परिमित तत्व विधि का विश्लेषण करते हैं। विश्लेषण मुख्य रूप से कई सहायक समस्याओं और हेल्महोल्ट्ज अपघटन के उपयोग पर निर्भर करता है। अध्याय 4 दूसरे क्रम डिरिचलेट सीमा नियंत्रण समस्या पर लागू असंतत गैलरकिन (डीजी) विधियों के एक विशिष्ट वर्ग के लिए एक प्राथमिकता और एक पश्चवर्ती त्रुटि अनुमान की व्युत्पत्ति प्रस्तुत करता है। विश्लेषण में एक महत्वपूर्ण कदम में एक अलग लागत कार्यात्मक का सावधानीपूर्वक चयन शामिल है जो बिलिनियर फॉर्म की सममित संपत्ति का लाभ उठाते हुए, निरंतर लागत कार्यात्मक का उचित अनुमान लगाता है। यह सावधानीपूर्वक चयन विश्लेषण की सटीकता और प्रभावशीलता सुनिश्चित करता है। अध्याय 5 में, हम नियंत्रण बाधित डिरिचलेट सीमा नियंत्रण समस्या की जांच में उतरते हैं, जो दूसरे क्रम के अण्डाकार आंशिक अंतर समीकरण (पीडीई) द्वारा शासित होती है। यह समस्या अध्याय 3 और अध्याय 4 में चर्चा की गई समस्या के विपरीत है, जहां शासी समीकरण दूसरे क्रम का अण्डाकार पीडीई बना हुआ है, लेकिन नियंत्रण चर पर कोई बाधा नहीं लगाई गई है। डीजी विधियों के सममित वर्ग का विश्लेषण प्राथमिक और पश्च त्रुटि अनुमान दोनों प्राप्त करने के लिए किया जाता है। अध्याय 6 में, चौथे क्रम डिरिचलेट सीमा नियंत्रण समस्या के लिए एक प्राथमिकता और एक पश्च त्रुटि विश्लेषण करने के लिए सममित आंतरिक दंड गैलेरकिन (एसआईपीजी) परिमित तत्व विधि का विश्लेषण किया गया है। जब दंड मापदंडों को उपयुक्त रूप से चुना जाता है, तो संपूर्ण चौथा क्रम एसआईपीजी परिमित तत्व विधि क्लासिक सी0–आंतरिक दंड विधि में भी कम हो जाती है। सी0–आंतरिक दंड विधि के लिए संबंधित त्रुटि अनुमान (प्रायोरी और पोस्टीरियर दोनों) प्राप्त किए जाते हैं। अध्याय 2 से अध्याय 6 तक प्रत्येक अध्याय के अंत में,

उनके संबंधित अध्यायों में स्थापित सैद्धांतिक निष्कर्षों का व्यावहारिक चित्रण प्रदान करने के लिए संख्यात्मक परिणाम प्रदर्शित किए जाते हैं। अध्याय 7 में, हम पूर्ण किए गए कार्य का सारांश प्रदान करते हैं और संभावित विस्तार का सुझाव देते हैं और प्रस्तुत कार्य के लिए भविष्य की दिशाएं प्रस्तावित करते हैं।

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Symbol	**Meaning**
$\mathcal{T}_h$	a regular triangulation of Ω,
T	a simplex of $\mathcal{T}_h$,
$\lvert T\rvert$	volume of an element T,
e	an edge or a face of $\mathcal{T}_h$,
$\lvert e\rvert$	length of an edge or a face e,
p	a node of an element T,
$\mathcal{V}_h$	set of all vertices in $\mathcal{T}_h$,
$\mathcal{V}_T$	set of vertices of the simplex T,
$\mathcal{V}_h^i$	set of all vertices in $\mathcal{T}_h$ that are in Ω,
$\mathcal{V}_h^b$	set of all vertices of $\mathcal{T}_h$ that are on $\partial\Omega$,
$\mathcal{V}_h^e$	set of all vertices lying on edge or face e,
$\mathcal{E}_h$	set of all edges or faces of $\mathcal{T}_h$,
$\mathcal{E}_h^i$	set of all interior edges or faces of $\mathcal{T}_h$,
$\mathcal{E}_h^b$	set of all edges or faces of $\mathcal{T}_h$ that are on $\partial\Omega$,
$\mathcal{E}_p$	set of all edges or faces connected to a given vertex p,
$\mathbb{P}_k(T)$	the space of polynomials of degree atmost k over element T,
ω_p	union of all elements sharing the node p,
ω_e	$\bigcup\limits_{e\subset\partial T} T$ for any edge or face e,
$\lvert\lvert\cdot\rvert\rvert_{m,p,D}$	$\lvert\lvert\cdot\rvert\rvert_{W^{m,p}(D)}$ where $m\in\mathbb{Z}\setminus\{0\}$ and $1\leq p\leq\infty$ and D $\subseteq\Omega$,
$\lvert\lvert\cdot\rvert\rvert_{m,D}$	$\lvert\lvert\cdot\rvert\rvert_{H^m(D)}$ where $m\in\mathbb{Z}\setminus\{0\}$ and $1\leq p\leq\infty$ and D $\subseteq\Omega$,
$\lvert\cdot\rvert_{m,p,D}$	the associated semi norm on $W^{m,p}(D)$ where $D\subseteq\Omega$,
$\lvert\cdot\rvert_{m,D}$	the associated semi norm on $H^m(D)$ where $D\subseteq\Omega$,

$\|\cdot\|_{1/2,\Gamma}$	the associated norm on $H^{1/2}(\Gamma)$ where $\Gamma = \partial\Omega$,
$\|\cdot\|_{-\frac{1}{2},\Gamma}$	the associated norm on $H^{-1/2}(\Gamma)$ where $\Gamma = \partial\Omega$,
$\|\cdot\|_{\infty,D}$	$\|\cdot\|_{L^\infty(D)}$, where $D \subseteq \Omega$,
$\|\cdot\|$	the associated norm on $L^2(\Omega)$,
$\|\cdot\|_{L^2(D)}$	the associated norm on $L^2(D)$ where $D \subsetneq \Omega$,
$(\cdot,\cdot)$	the associated inner product on $L^2(\Omega)$,
$\delta_{i,j}$	the Kronecker's Delta i.e., $\delta_{i,j} = 1$ if $i = j$ and 0 elsewhere,
$\boldsymbol{v}'$	the transpose of vector valued function $\boldsymbol{v}$,
$L^p(\Omega)$	Lebesgue spaces, $1 \leq p \leq \infty$,
$W^{m,p}(\Omega)$	Sobolev spaces, $1 \leq p \leq \infty$,
$H^m(\Omega),\ H_0^m(\Omega)$	Sobolev spaces $W^{m,p}(\Omega)$ for $p = 2$,
$H^m(\Omega, \mathcal{T}_h)$	piece-wise Sobolev space of order $m \in \mathbb{N}$,
$C^m(\Omega)$	the space of m times continuously differentiable functions,
$D(\Omega)$	the space of infinitely differentiable functions having compact support in Ω,
C	generic positive constant, independent of perturbation and mesh parameters,
$X \lesssim Y$	$X \leq CY$ for some positive constant C, which is independent of the mesh size h,
$C_c^m(\Omega)$	the space of m times continuously differentiable functions with compact support in Ω,
$meas(A)$	measure of the set $A \subseteq \bar{\Omega}$,
$supp(f)$	support of function f.