A STUDY OF DIFFUSION AND ENTRANCE FLOW PROBLEMS RELEVANT TO THE BIOLOGICAL SYSTEM

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ON

A STUDY OF DIFFUSION AND ENTRANCE FLOW PROBLEMS RELEVANT TO THE BIOLOGICAL SYSTEM

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SYNOPSIS

The present thesis consists of four Chapters.

Chapter I presents a brief review of the previous studies mainly related to the present work.

Chapter II deals with the facilitated diffusional transport of O_2 and CO_2 through a medium (such as the lung capillaries) which contains protein (haemoglobin in the present analysis), and in the last two Chapters, some problems on steady and unsteady entrance flow of a viscous incompressible fluid are discussed.

In the first problem of Chapter II, a simple mathematical model for the facilitated diffusional transport of O_2 and CO_2 across a membrane containing haemoglobin has been discussed. This study is based on the experimental and analytic works of Wittenberg (1971) and Murray (1974) who consider the transport of oxygen in the presence of haemoglobin and myoglobin. An asymptotic solution to the basic diffusion equations including chemical reactions between the species oxygen-haemoglobin and carbondioxide-haemoglobin has been given. The available physiological data e.g., the concentration of the gases in the alveolar region and in the capillaries, diffusion coefficients of haemoglobin, carbondioxide and oxygen in the membrane, and the rate of forward and reverse reactions for oxygen and carbondioxide respectively with haemoglobin, have been used in the analysis.

A relationship has been observed between the saturation functions for O_2 and CO_2 which is in agreement with the well known Bohr effect and Haldane effect, Guyton (1971). It has been concluded that to the first order of approximation, the concentration of the dissolved CO_2 and corresponding saturation function are found to vary linearly with the distance while this is not the case for O_2.

The motivation for the second problem of Chapter II is provided by the transport of O_2 and CO_2 in the pulmonary capillaries. Hence we consider the diffusion of O_2 and CO_2 in the presence of haemoglobin when the species have attained equilibration in the capillaries while the governing equations in the two problems would be the same (system of nonlinear second order ordinary differential equations), the main difference would be in the boundary conditions. We have obtained solution in the closed form in this problem.

It has been concluded that at equilibration, most of the haemoglobin has combined with the O_2 and very little free haemoglobin is left. Further, the amount of carbaminohaemoglobin ($HbCO_2$) at equilibration is negligibly small.

Chapter III contains two problems on steady development of laminar axisymmetric flow of an incompressible viscous fluid from the entry to the fully developed region in a straight pipe of radius a.

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The development of the flow has been divided into three regions (i) inviscid core, (ii) boundary layer and (iii) the downstream region. The analysis is based on the method of matched asymptotic expansions.

In the first problem, it is assumed that the fluid enters the pipe with a constant axial velocity $\bar{w}_0$ and a constant swirl velocity $\frac{A\bar{w}_0 r'}{a}$, where r' is the radial co-ordinate and A is the swirl constant. Solution in different regions has been obtained for various values of A. It is found that the solution for the developing flow breaks down when the axial distance $x = O(Re)$. This non-uniformity for large x has been resolved by introducing a new streamwise co-ordinate $\xi = \frac{x}{Re}$, which reduces the analysis to the problem discussed by M. Kiya, Shoichiro Fukusako and Mikio Arie (1971).

In the second problem uniform entry condition is considered and the results for flow near the entry are found to be in good agreement with the numerical solution of full Navier-Stokes equations obtained by Friedmann, Gillis and Firon (1968). The solution for developing flow breaks down for streamwise distance $x = O(Re)$, where $Re = \frac{a\bar{w}_0}{\nu}$, $\bar{w}_0$ being the characteristic velocity at the entry. Accordingly, we introduce a stretched streamwise co-ordinate $\xi = \frac{x}{Re}$ in terms of which the new problem is set up to determine the downstream flow which is found to coincide with the problem discussed by Hornbeck (1964).

It has been shown that as the swirl velocity increases at the entry, the axial velocity tends to increase in the core region close to the entrance but further downstream, it tends to increase near the wall and decreases near the pipe axis. It is found that the axial skin friction is not influenced by the swirl parameter near the entrance region.

In Chapter IV, two problems on unsteady development of laminar axisymmetric flow of an incompressible viscous fluid from the entry to the downstream region in a straight pipe and a curved pipe, respectively have been studied by using the method of matched asymptotic expansions.

In the first problem, the entry condition has been assumed as,

$$w(t') = \bar{w}_0 \, Q(\omega t'),$$

where t' is the time, $\frac{1}{\omega}$ is a typical time scale of the motion and $\bar{w}_0$ is the characteristic velocity at the entry. The only restriction on $Q(\omega t')$ is that it should always be positive (no reversal of the free stream) i.e.,

$$Q(\omega t') = 1 + \alpha_1 \sin \omega t', \quad \text{where } 0 \leq \alpha_1 < 1.$$

The development of the flow has been divided into three regions. It is found that the solution for the developing flow breaks down when the axial distance $x = O(\epsilon^{-1})$, where $\epsilon = \frac{a\omega}{\bar{w}_0}$ and a is the radius of the pipe.

This non-uniformity for large x has been resolved by introducing a new streamwise co-ordinate $\xi = \epsilon x$, which reduces the analysis to the problem discussed by T.J. Pedley (1972) for large values of x. The effect of the outer boundary layer on the core flow has also been obtained for the downstream region.

It has been shown that with an increase in the value of α_1, the axial velocity tends to increase near the core region close to the entrance, but further downstream, this effect becomes negligible. It is found that the upstream axial velocity is maximum near $t = \frac{\pi}{2}$ and minimum near $t = \frac{3\pi}{2}$, where $t = \omega t'$.

The second problem deals with the entry flow in the ascending aorta. The analysis follows closely the approach made by Singh (1974) for the steady entrance flow in a curved pipe. The entrance velocity is assumed as

$$w = \frac{Q(t)}{1 + \delta r \cos\alpha} \; , \quad Q(t) \geq 0$$

where t is the non-dimensional time, δ denotes the curvature effect, r is the radial co-ordinate and α is the azimuthal angle. The available physiological data for the circulation system of human are taken into account for estimating the orders of magnitude of the parameters arising in the dimensionless equations of motion. Boundary layer analysis is applied to determine the effect of curvature and adverse pressure gradient (associated with the primary flow) on the wall shear.

The study shows how the negative wall shear and back flow near the wall develop during the decelerating phase of the cycle as the boundary layer grows. Also it is shown that how the increasing effect of the secondary flow due to curvature draws off slower moving fluid azimuthally from the outer bend to the inner bend and induces a cross-flow of faster moving fluid from the inner bend to the outer bend which results in thinning of the boundary layer at the outer bend and its thickenning at the inner bend. This implies increased wall shear at the outer bend in comparison to the inner bend as the flow develops further downstream which is in contrast with the initial stages of the motion near the entrance where the higher wall shear occurs at the inner bend due to the external flow and geometric factors. The analysis shows that the displacement effect of the boundary layer on the core is not significant due to the fact that the boundary layer remains thin about 1/10 th of the tube diameter.
