

# ENTROPY STABLE SCHEMES FOR RELATIVISTIC FLOWS

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# ENTROPY STABLE SCHEMES FOR RELATIVISTIC FLOWS

by

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Department of Mathematics

*Submitted*

*in fulfillment of the requirements of the degree of Doctor of Philosophy*

*to the*



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*Dedicated to my parents.*



# Certificate

This is to certify that the thesis entitled **Entropy Stable Schemes for Relativistic Flows** submitted by **Mr. Deepak Bhoriya** to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

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# Abstract

Relativistic fluid and plasma flows are fundamental for understanding many astrophysical applications, e.g., supernova, extragalactic jets, etc. Most of these phenomena have fluid or plasma moving close to the speed of light. Hence, the relativistic effects need to be considered as standard fluid or plasma models are not able to capture relativistic effects. For the applications where electromagnetic fields are not important, only fluid parts are considered, and they are often modeled via equations of relativistic hydrodynamics (RHD). The RHD equations are the simplest relativistic fluid model, and they are a system of hyperbolic conservation laws with nonlinear flux.

Due to the nonlinearity of the flux function, solutions develop discontinuities, even when initial conditions are smooth. Hence, weak solutions need to be considered. However, weak solutions are not unique, and a non-physical solution can also be a weak solution. To overcome this, an additional criterion in the form of entropy condition is considered. In addition, the entropy criteria provides a non-linear stability estimate for the solution. So, it is expected that any numerical method should produce solutions that satisfy entropy condition.

The first contribution of the thesis is to design high-order finite-difference numerical schemes for RHD equations, which are entropy stable. Due to the presence of the Lorentz factor, the conversion from conservative to primitive variables is not straightforward. We need to solve a nonlinear equation to achieve this conversion. For the entropy condition, we first present the entropy inequality for the continuous model. To obtain semi-discrete numerical schemes which satisfy entropy conditions at the discrete level, we first construct the entropy conservative schemes, followed by the entropy stable schemes using the higher-order entropy diffusion operators. We establish fully discrete schemes using the high-order accurate Strong Stability Preserving Runge Kutta (SSP-RK) time-stepping methods. We apply the proposed schemes to several test cases in one and two dimensions to demonstrate the stability of the proposed

schemes.

In the second contribution of the thesis, we design high-order nodal Discontinuous Galerkin (DG) methods, which are entropy stable. The schemes are based on the Summation By Parts (SBP) property of the Gauss-Lobatto quadratures and the lagrangian nodal basis. Coupling this with the entropy conservative numerical flux used for finite-difference schemes and entropy stable Lax-Friedrich flux, we prove that the proposed schemes are entropy stable. We present second, third, and fourth-order schemes, which are then tested on various test cases.

In several astrophysical applications, electric and magnetic fields need to be considered. Often these are modeled via equations of relativistic magnetohydrodynamics (RMHD). However, equations of RMHD are derived using several assumptions, which make them unsuitable for some applications. For those applications, two-fluid relativistic plasma flow equations are more suitable. In this work, we design entropy stable numerical schemes for the two-fluid relativistic plasma flow model.

In the third contribution of the thesis, we propose high-order finite-difference entropy stable schemes for the two-fluid relativistic plasma flow equations. This is achieved by exploiting the structure of the equations, which consists of three independent flux components. The first two components describe the ion and electron flows, which are modeled using the relativistic hydrodynamics equation. The third component is Maxwell's equations, which are linear systems. The coupling of the ion and electron flows, and electromagnetic fields is via source terms only. Furthermore, we also show that the source terms do not affect the entropy evolution. To design semi-discrete entropy stable schemes, we extend the RHD entropy stable schemes to three dimensions. This is then coupled with entropy stable discretization of the Maxwell's equations. Finally, we use SSP-RK schemes to discretize in time. We also propose ARK-IMEX schemes to treat the stiff source terms. The resulting nonlinear set of algebraic equations is local (at each discretization point). These equations are solved using the Newton's method, which results in an efficient method. The proposed schemes are then tested using various test problems to demonstrate their stability, accuracy, and efficiency.

In the final contribution of this thesis, we design entropy stable Discontinuous Galerkin schemes for two-fluid relativistic plasma flow equations. The proposed schemes are based on the Gauss-Lobatto quadrature rule, which has summation by parts (SBP) property. We exploit the structure of the equations having the flux with three independent parts, which are coupled via stiff nonlinear source terms. We design an entropy stable DG scheme for each part. This coupled with the fact that the source terms do not affect entropy, results in an entropy stable

scheme for the complete system. The proposed schemes are then tested on a variety of test problems to demonstrate their accuracy and stability.



# सार

कई खगोलीय अनुप्रयोगों, जैसे, सुपरनोवा, एक्सट्रैगैलेक्टिक जेट, आदि को समझने के लिए सापेक्ष द्रव और प्लाज्मा प्रवाह मौलिक हैं। इनमें से अधिकांश घटनाओं में द्रव या प्लाज्मा प्रकाश की गति के करीब चलते हैं। इसलिए, सापेक्षतावादी प्रभावों को मानक द्रव के रूप में माना जाना चाहिए या प्लाज्मा मॉडल सापेक्षतावादी प्रभावों को पकड़ने में सक्षम नहीं हैं। उन अनुप्रयोगों के लिए जहां विद्युत चुम्बकीय क्षेत्र महत्वपूर्ण नहीं हैं, केवल द्रव भागों पर विचार किया जाता है, और उन्हें अक्सर सापेक्षतावादी हाइड्रोडायनामिक्स (आरएचडी) के समीकरणों के माध्यम से तैयार किया जाता है। आरएचडी समीकरण सबसे सरल सापेक्षतावादी द्रव मॉडल हैं, और वे गैर-रेखीय प्रवाह के साथ अतिपरवलयिक संरक्षण कानूनों की एक प्रणाली हैं।

फ्लक्स फंक्शन की गैर-रैखिकता के कारण, प्रारंभिक स्थितियां सुचारू होने पर भी, समाधान असंततता विकसित करते हैं। इसलिए, कमजोर समाधानों पर विचार करने की आवश्यकता है। हालांकि, कमजोर समाधान अद्वितीय नहीं हैं, और एक गैर-भौतिक समाधान भी एक कमजोर समाधान हो सकता है। इसे दूर करने के लिए, एन्ट्रापी स्थिति के रूप में एक अतिरिक्त मानदंड पर विचार किया जाता है। इसके अलावा, एन्ट्रापी मानदंड समाधान के लिए एक गैर-रैखिक स्थिरता अनुमान प्रदान करता है। इसलिए, यह अपेक्षा की जाती है कि किसी भी संख्यात्मक विधि से ऐसे समाधान उत्पन्न हों जो एन्ट्रापी की स्थिति को संतुष्ट करते हों।

थीसिस का पहला योगदान आरएचडी समीकरणों के लिए उच्च-क्रम परिमित-अंतर संख्यात्मक योजनाओं को डिजाइन करना है, जो एन्ट्रापी स्थिर हैं। लोरेन्ज़ कारक की उपस्थिति के कारण, रूढ़िवादी से आदिम चर में रूपांतरण सीधा नहीं है। इस रूपांतरण को प्राप्त करने के लिए हमें एक गैर-रेखीय समीकरण को हल करने की आवश्यकता है। एन्ट्रापी स्थिति के लिए, हम पहले निरंतर मॉडल के लिए एन्ट्रापी असमानता प्रस्तुत करते हैं। अर्ध-असतत संख्यात्मक योजनाएं प्राप्त करने के लिए जो असतत स्तर पर एन्ट्रापी की स्थिति को संतुष्ट करती हैं, हम पहले एन्ट्रापी रूढ़िवादी योजनाओं का निर्माण करते हैं, इसके बाद उच्च-क्रम एन्ट्रापी प्रसार ऑपरेटर्स का उपयोग करके एन्ट्रापी स्थिर योजनाओं का निर्माण करते हैं। हम हाई-ऑर्डर सटीक स्ट्रॉन्ग स्टेबिलिटी प्रिजर्विंग रनज कुट्टा (एसएसपी-आरके) टाइम-स्टेपिंग विधियों का उपयोग करके पूरी तरह से असतत योजनाएं स्थापित करते हैं। हम प्रस्तावित योजनाओं की स्थिरता को प्रदर्शित करने के लिए एक और दो आयामों में कई परीक्षण मामलों में प्रस्तावित योजनाओं को लागू करते हैं।

थीसिस के दूसरे योगदान में, हम उच्च-क्रम नोडल डिसकंटिन्यूअस गैलेरकिन (डीजी) विधियों को डिजाइन करते हैं, जो एन्ट्रापी स्थिर हैं। योजनाएं गॉस-लोबेटो चतुर्भुज और लैग्रेंजियन नोडल आधार की समेशन बाय पार्ट्स (एसबीपी) संपत्ति पर

आधारित हैं। इसे परिमित-अंतर योजनाओं और एन्ट्रॉपी स्थिर लैक्स-फ्रेडरिक प्रवाह के लिए उपयोग किए जाने वाले एन्ट्रापी रूढ़िवादी संख्यात्मक प्रवाह के साथ जोड़कर, हम साबित करते हैं कि प्रस्तावित योजनाएं एन्ट्रापी स्थिर हैं। हम दूसरे, तीसरे और चौथे क्रम की योजनाएं प्रस्तुत करते हैं, जिनका परीक्षण विभिन्न परीक्षण मामलों पर किया जाता है।

कई खगोलभौतिकीय अनुप्रयोगों में, विद्युत और चुंबकीय क्षेत्रों पर विचार करने की आवश्यकता होती है। अक्सर इन्हें सापेक्षतावादी मैग्नेटोहाइड्रोडायनामिक्स (आरएमएचडी) के समीकरणों के माध्यम से तैयार किया जाता है। हालाँकि, आर एम एच डी के समीकरण कई मान्यताओं का उपयोग करके प्राप्त किए जाते हैं, जो उन्हें कुछ अनुप्रयोगों के लिए अनुपयुक्त बनाते हैं। उन अनुप्रयोगों के लिए, दो-द्रव सापेक्षतावादी प्लाज्मा प्रवाह समीकरण अधिक उपयुक्त हैं। इस काम में, हम दो-द्रव सापेक्षतावादी प्लाज्मा प्रवाह मॉडल के लिए एन्ट्रापी स्थिर संख्यात्मक योजनाएं तैयार करते हैं।

थीसिस के तीसरे योगदान में, हम दो-तरल सापेक्षतावादी प्लाज्मा प्रवाह समीकरणों के लिए उच्च-क्रम परिमित-अंतर एन्ट्रापी स्थिर योजनाओं का प्रस्ताव करते हैं। यह समीकरणों की संरचना का उपयोग करके प्राप्त किया जाता है, जिसमें तीन स्वतंत्र प्रवाह घटक होते हैं। पहले दो घटक आयन और इलेक्ट्रॉन प्रवाह का वर्णन करते हैं, जो कि सापेक्षतावादी हाइड्रोडायनामिक्स समीकरण का उपयोग करके बनाए गए हैं। तीसरा घटक मैक्सवेल के समीकरण हैं, जो रैखिक प्रणाली हैं। आयन और इलेक्ट्रॉन प्रवाह का युग्मन, और विद्युत चुंबकीय क्षेत्र केवल स्रोत शब्दों के माध्यम से होता है। इसके अलावा, हम यह भी दिखाते हैं कि स्रोत शब्द एन्ट्रापी विकास को प्रभावित नहीं करते हैं। अर्ध-असतत एन्ट्रापी स्थिर योजनाओं को डिजाइन करने के लिए, हम आरएमएचडी एन्ट्रापी स्थिर योजनाओं को तीन आयामों तक बढ़ाते हैं। इसके बाद मैक्सवेल के समीकरणों के एन्ट्रॉपी स्थिर विवेकीकरण के साथ जोड़ा जाता है। अंत में, हम समय पर विवेक के लिए एसएसपी-आरके योजनाओं का उपयोग करते हैं। हम कठोर स्रोत शर्तों का इलाज करने के लिए आरक-ईमेक्स योजनाओं का भी प्रस्ताव करते हैं। बीजीय समीकरणों का परिणामी अरैखिक समुच्चय स्थानीय होता है (प्रत्येक विच्छेदन बिंदु पर)। इन समीकरणों को न्यूटन की विधि का उपयोग करके हल किया जाता है, जिसके परिणामस्वरूप एक कुशल विधि प्राप्त होती है। प्रस्तावित योजनाओं का परीक्षण उनकी स्थिरता, सटीकता और दक्षता प्रदर्शित करने के लिए विभिन्न परीक्षण समस्याओं का उपयोग करके किया जाता है।

इस थीसिस के अंतिम योगदान में, हम दो-तरल सापेक्षतावादी प्लाज्मा प्रवाह समीकरणों के लिए एन्ट्रापी स्थिर असंतत गैलेरिकन योजनाओं को डिजाइन करते हैं। प्रस्तावित योजनाएं गॉस-लोबेटो क्वाड्रेचर नियम पर आधारित हैं, जिसमें भागों (एसबीपी) संपत्ति का योग है। हम तीन स्वतंत्र भागों के साथ प्लक्स वाले समीकरणों की संरचना का फायदा उठाते हैं, जो कठोर गैर-रेखीय स्रोत शब्दों के माध्यम से युग्मित होते हैं। हम प्रत्येक भाग के लिए एक एन्ट्रापी स्थिर डीजी योजना तैयार करते हैं। यह इस तथ्य के साथ युग्मित है कि स्रोत शब्द एन्ट्रापी को प्रभावित नहीं करते हैं, जिसके परिणामस्वरूप संपूर्ण प्रणाली के लिए एक एन्ट्रापी स्थिर योजना होती है। प्रस्तावित योजनाओं का परीक्षण उनकी सटीकता और स्थिरता को प्रदर्शित करने के लिए विभिन्न परीक्षण समस्याओं पर किया जाता है।

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# List of Symbols

|  |  |
|--|--|
| $\mathbb{R}$   | Set of Real numbers  |
| $\mathbb{Z}$   | Set of Integers  |
| $\mathbb{N}$   | Set of Natural numbers   |
| $\mathbb{R}^n$   | $n$ -dimensional Cartesian Product of $\mathbb{R}$   |
| $\in$  | Belongs to   |
| $\subset$  | Subset   |
| $\forall$  | For all  |
| $(\cdot)^\top$   | Transpose of a vector (or a matrix)  |
| $\nabla \cdot \mathbf{F}$                              | Gradient of a vector $\mathbf{F}$  |
| $\Delta \mathbf{U}$                                    | Hessian of a vector $\mathbf{U}$   |
| $\mathbf{C}^1(D)$                                      | Space of differentiable functions whose derivative is continuous   |
| $\mathbf{L}_{loc}^\infty(D)$                           | Space of Locally almost everywhere bounded integrable function over a domain $D$                             |
| $\mathbf{C}_0^\infty(D)$                               | Space of continuously differentiable functions with compact support  |
| $n_c$  | Number of unknown variables in a balance law   |
| $n_d$  | Dimension of a balance law   |
| $\mathbf{U}$   | Vector of conservative variables   |
| $\mathbf{f}^j(\mathbf{U})$                             | Exact continuous fluxes  |
| $\mathbf{s}(\mathbf{U})$                               | The source vector  |
| $j$  | Set of eigenvalues for the jacobian matrix $\frac{\partial \mathbf{f}^j}{\partial \mathbf{U}}$               |
| $\mathbf{R}^j$   | The matrix of right eigenvectors for the jacobian matrix $\frac{\partial \mathbf{f}^j}{\partial \mathbf{U}}$ |
| $(\mathcal{U}(\mathbf{U}), \mathcal{F}^j(\mathbf{U}))$ | Entropy-entropy flux pair  |
| $\mathcal{V}(\mathbf{U})$                              | Entropy variable   |