

**ELLIPTIC EQUATIONS AND NON-LOCAL ELLIPTIC  
EQUATIONS WITH SIGN-CHANGING NONLINEARITIES**

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INDIAN INSTITUTE OF TECHNOLOGY DELHI  
JANUARY 2015**

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**ELLIPTIC EQUATIONS AND NON-LOCAL  
ELLIPTIC EQUATIONS WITH SIGN-CHANGING  
NONLINEARITIES**

by

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Submitted

in fulfillment of the requirements of the degree of  
**Doctor of Philosophy**

*to the*



**Indian Institute of Technology Delhi**

**JANUARY 2015**

# Certificate

This is to certify that the thesis entitled “**Elliptic equations and Non-local elliptic equations with sign-changing nonlinearities**” submitted by **Mrs. Sarika Goyal** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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# Acknowledgements

*It is with immense gratitude that I acknowledge all those people without whom this thesis might not have been written.*

*First and foremost, I owe my deepest gratitude to my supervisor Prof. K. Sreenadh for his expert supervision and continuous support. I would like to express the deepest appreciation to Prof. K. Sreenadh, for his patience, motivation, valuable suggestions, and immense knowledge. I appreciate all his contributions of time, ideas, and funding to make my Ph.D. experience productive and stimulating. The joy and enthusiasm he has for his research was contagious and motivational for me throughout the Ph.D. pursuit.*

*I am thankful to IIT Delhi and all the faculty members of the Department of Mathematics, IIT Delhi for providing me the necessary facilities for smooth completion of my research. I would like to thank my SRC (Student Research Committee) members Prof. Niladri Chatterjee, Dr. V.V.K. Srinivas, and Prof. Sanjeev Sangi for always giving me time for all the official presentations and their valuable feedback. I would like to mention a special thank to Dr. Ratikanta Panda of Delhi University for taking pain of travelling to IIT Delhi and judging my progress whenever required.*

*Also I would like to acknowledge University Grants Commission for providing me financial assistance throughout my research work.*

*My time at IIT Delhi was made enjoyable in large part due to my friends Amita, Arti and Ishanki who became a part of my life. A special thanks to my senior Sweta Tiwari for always being there when needed. I would like to thank my colleagues Varsha, Kavita, Manisha, Arti, Anju, Sumit, Dinesh and sudhakar for helping me during my teaching assistantship and providing me a comfortable environment. Interactions with my juniors Parwan have always taught me something new and thus I*

*owe them for discussing their research problems with me.*

*Also I share the credit of my work with all my teachers who have taught me at some level of my academic journey specially mentioning the name of my schooldays maths teacher Rajni mam and Tej Singh Sir for making me believe that Mathematics is a fun. I would also like to mention a special thank to my school teacher Hari Kishan Sir who encouraged me to come Delhi for higher study.*

*Last but not the least I am indebted forever to my entire family for the faith, encouragement, blessings and love they showered at me. I thank my parents Mr. Vijay Kumar Goyal, Mrs. Saroj Goyal for not only giving me wings but also opportunity to fly high. I owe my deepest gratitude to my sisters Nisha Goyal, Monika Goyal and brother Bhupesh Goyal for their affection and good wishes. A special thank goes to the my new family members Shri Naresh Kumar Gupta, Mrs. Uma Gupta and my sister in laws Dipti Goyal, Mohini Bansal and Shanu Gupta for always cheering me up. And most of all for my loving, supportive, encouraging, and patient husband Mr. Varun Gupta whose faithful support during the final stages of my Ph.D. is so appreciated. Thank you.*

*New Delhi*

*January 2015*

*Sarika Goyal*

# Abstract

In this thesis, we study the existence, non-existence and multiplicity of positive solutions of elliptic equations and non-local elliptic equations with sign-changing nonlinearities. In the first chapter we give the brief survey and preliminary results used in subsequent chapters.

In the second chapter, we study the existence and multiplicity of solutions of the singular  $n$ -Laplace equation:

$$(P_\lambda) \quad \left\{ \begin{array}{l} -\Delta_n u + V(x)|u|^{n-2}u = \frac{|u|^p}{|x|^\beta} e^{|u|^{\frac{n}{n-1}}} + \lambda h(x)u^q, \quad u > 0 \text{ in } \mathbb{R}^n, \end{array} \right.$$

where  $n \geq 2$ ,  $0 < q < n - 1 < p + 1$ ,  $\beta \in [0, n)$ ,  $\lambda > 0$ , and  $h \geq 0$  in  $\mathbb{R}^n$ . Using the nature of the Nehari manifold and fibering maps associated with the Euler functional, we prove that there exists  $\lambda_0$  such that for  $\lambda \in (0, \lambda_0)$ , the problem admits at least two positive solutions. We also show that when  $h(x) > 0$ , there exists  $\lambda^0$  such that  $(P_\lambda)$  has no solution for  $\lambda > \lambda^0$ .

In the third chapter, we study the following  $n$ -Laplace equation:

$$(P_{\lambda, V, h}) \quad \left\{ \begin{array}{l} -\Delta_n u + V(x)|u|^{n-2}u = \lambda h(x)|u|^{q-1}u + u|u|^p e^{|u|^\beta} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \end{array} \right.$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n \geq 2$ ,  $0 < q < n - 1 < p + 1$ ,  $\beta \in (1, \frac{n}{n-1}]$  and  $\lambda > 0$ . By minimization on the suitable subset of the Nehari manifold using the fiber maps, we find conditions on  $V$ ,  $h$  that yields the existence and multiplicity of non-negative solutions when  $V$ ,  $h$  are sign-changing and unbounded functions.

In the fourth chapter, we extend the results of the previous chapter to the quasi polyharmonic operators. We consider the following equation:

$$\Delta_{\frac{m}{m}}^m u = \lambda h(x)|u|^{q-1}u + u|u|^p e^{|u|^\beta} \text{ in } \Omega, \quad u = \nabla u = \dots = \nabla^{m-1}u = 0 \text{ on } \partial\Omega,$$

where  $\Omega$  is a bounded domain with smooth boundary in  $\mathbb{R}^n$ ,  $n \geq 2m \geq 2$ ,  $0 < q < \frac{n}{m} - 1 < p + 1$ ,  $\beta \in (1, \frac{n}{n-m}]$ ,  $\lambda > 0$  and  $h$  is sign-changing and unbounded function. Using the Nehari manifold and fibering maps, we show the existence and multiplicity of solutions. In this chapter, we also study the existence result for the problem with superlinear type nonlinearity.

In the fifth chapter, we study the following  $n$ -Kirchhoff equation

$$(\mathcal{M}) \quad m \left( \int_{\Omega} |\nabla u|^n dx \right) \Delta_n u = f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where  $\Omega$  is a bounded domain with smooth boundary in  $\mathbb{R}^n$ ,  $m : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a continuous function. Later we also study the existence of multiple solutions of the kirchhoff equation involving  $n$ -Laplacian with exponential nonlinearity and sign-changing weight function by Nehari manifold and fibering map analysis.

In the sixth chapter, we study the Fučík spectrum of non-local operator which is defined as the set of all  $(a, b) \in \mathbb{R}^2$  such that

$$-2 \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x - y|^{n+p\alpha}} dy = a(u^+)^{p-1} - b(u^-)^{p-1} \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

has a non-trivial solution  $u$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with Lipschitz boundary,  $n > p\alpha$ ,  $\alpha \in (0, 1)$  and  $p \geq 2$ . The existence of a first non-trivial curve  $\mathcal{C}$  of this spectrum, some properties of this curve  $\mathcal{C}$ , e.g. Lipschitz continuous, strictly decreasing and asymptotic behavior are studied. At the end, we also study a non-resonance problem with respect to the Fučík spectrum.

In seventh chapter, we study the existence and multiplicity of non-negative solutions of non-local equation with convex-concave nonlinearity  $f_\lambda(x, u) = \lambda h(x)|u|^{q-1}u + b(x)|u|^{r-1}u$ , where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $n > p\alpha$ ,  $\alpha \in (0, 1)$ ,  $0 < q \leq p - 1 <$

$r < \frac{np}{n-p\alpha} - 1$ ,  $\lambda > 0$  and  $h, b$  are sign-changing bounded functions. First we study the case  $q < p - 1$  and show existence and multiplicity of solutions by minimization on the suitable subset of Nehari manifold using the fibering maps. We find that there exists  $\lambda_0$  such that for  $\lambda \in (0, \lambda_0)$ , it has at least two non-negative solutions. Next we study the case  $q = p - 1$  and  $h(x) = 1$ .

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