ELLIPTIC EQUATIONS AND NON-LOCAL ELLIPTIC EQUATIONS WITH SIGN-CHANGING NONLINEARITIES

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by

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Certificate

This is to certify that the thesis entitled **"Elliptic equations and Non-local elliptic equations with sign-changing nonlinearities"** submitted by **Mrs. Sarika Goyal** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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Abstract

In this thesis, we study the existence, non-existence and multiplicity of positive solutions of elliptic equations and non-local elliptic equations with sign-changing nonlinearities. In the first chapter we give the brief survey and preliminary results used in subsequent chapters.

In the second chapter, we study the existence and multiplicity of solutions of the singular n-Laplace equation:

$$(P_\lambda) \quad \left\{ \quad -\Delta_n u + V(x)|u|^{n-2}u = \frac{u|u|^p}{|x|^\beta}\, e^{|u|^{\frac{n}{n-1}}} + \lambda h(x)u^q,\ u > 0 \text{ in } \mathbb{R}^n, \right.$$

where $n \geq 2$, $0 < q < n-1 < p+1$, $\beta \in [0, n)$, $\lambda > 0$, and $h \geq 0$ in $\mathbb{R}^n$. Using the nature of the Nehari manifold and fibering maps associated with the Euler functional, we prove that there exists λ_0 such that for $\lambda \in (0, \lambda_0)$, the problem admits at least two positive solutions. We also show that when $h(x) > 0$, there exists λ^0 such that (P_λ) has no solution for $\lambda > \lambda^0$.

In the third chapter, we study the following n-Laplace equation:

$$(P_{\lambda,V,h}) \quad \left\{ -\Delta_n u + V(x)|u|^{n-2}u = \lambda h(x)|u|^{q-1}u + u|u|^p\, e^{|u|^\beta} \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \right.$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $0 < q < n-1 < p+1$, $\beta \in \left(1, \frac{n}{n-1}\right]$ and $\lambda > 0$. By minimization on the suitable subset of the Nehari manifold using the fiber maps, we find conditions on V, h that yields the existence and multiplicity of non-negative solutions when V, h are sign-changing and unbounded functions.

In the fourth chapter, we extend the results of the previous chapter to the quasi polyharmonic operators. We consider the following equation:

$$\Delta^m_{\frac{n}{m}} u = \lambda h(x)|u|^{q-1}u + u|u|^p \, e^{|u|^\beta} \text{ in } \Omega, \quad u = \nabla u = \cdots = \nabla^{m-1}u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, $n \geq 2m \geq 2$, $0 < q < \frac{n}{m} - 1 < p + 1$, $\beta \in \left(1, \frac{n}{n-m}\right]$, $\lambda > 0$ and h is sign-changing and unbounded function. Using the Nehari manifold and fibering maps, we show the existence and multiplicity of solutions. In this chapter, we also study the existence result for the problem with superlinear type nonlinearity.

In the fifth chapter, we study the following $n-$Kirchhoff equation

$$(\mathcal{M}) \qquad m\left(\int_\Omega |\nabla u|^n dx\right) \Delta_n u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^n$, $m : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function. Later we also study the existence of multiple solutions of the kirchhoff equation involving $n-$Laplacian with exponential nonlinearity and sign-changing weight function by Nehari manifold and fibering map analysis.

In the sixth chapter, we study the Fučik spectrum of non-local operator which is defined as the set of all $(a,b) \in \mathbb{R}^2$ such that

$$-2\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|x-y|^{n+p\alpha}} dy = a(u^+)^{p-1} - b(u^-)^{p-1} \text{ in } \Omega,\ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

has a non-trivial solution u, where Ω is a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $n > p\alpha$, $\alpha \in (0,1)$ and $p \geq 2$. The existence of a first non-trivial curve $\mathcal{C}$ of this spectrum, some properties of this curve $\mathcal{C}$, e.g. Lipschitz continuous, strictly decreasing and asymptotic behavior are studied. At the end, we also study a non-resonance problem with respect to the Fučik spectrum.

In seventh chapter, we study the existence and multiplicity of non-negative solutions of non-local equation with convex-concave nonlinearity $f_\lambda(x,u) = \lambda h(x)|u|^{q-1}u + b(x)|u|^{r-1}u$, where Ω is a bounded domain in $\mathbb{R}^n$, $n > p\alpha$, $\alpha \in (0,1)$, $0 < q \leq p-1 <$

$r < \frac{np}{n-p\alpha} - 1$, $\lambda > 0$ and h, b are sign-changing bounded functions. First we study the case $q < p-1$ and show existence and multiplicity of solutions by minimization on the suitable subset of Nehari manifold using the fibering maps. We find that there exists λ_0 such that for $\lambda \in (0, \lambda_0)$, it has at least two non-negative solutions. Next we study the case $q = p-1$ and $h(x) = 1$.

Contents