

**ON THE IRREDUCIBILITY OF GENERALIZED LAGUERRE
POLYNOMIALS AND THEIR GALOIS GROUPS**

ANKITA JINDAL



**DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
JUNE 2019**

ON THE IRREDUCIBILITY OF GENERALIZED LAGUERRE POLYNOMIALS AND THEIR GALOIS GROUPS

by
ANKITA JINDAL

Department of Mathematics

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



Indian Institute of Technology Delhi
JUNE 2019

Dedicated to
My Family

Certificate

This is to certify that the thesis entitled “**On the Irreducibility of Generalized Laguerre Polynomials and their Galois Groups**” submitted by “**Ms. Ankita Jindal**” to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi

June 2019

Dr. Ritumoni Sarma

Associate Professor

Department of Mathematics

Indian Institute of Technology Delhi

Acknowledgements

It gives me great pleasure to express my sincere gratitude to my supervisor, Prof. Ritumoni Sarma, for his constant encouragement and the freedom he gave me while working on my thesis. His ardent dedication to perfection, exactness and completeness inspired me to write this thesis as nice as possible. His suggestions have helped me to improve the quality of this thesis. I would like to give my heartily thanks to him for being my supervisor.

I would like to take this opportunity to thank Prof. Shanta Laishram, Associate Professor, ISI Delhi, for agreeing to guide me as an unofficial thesis supervisor and for his expert guidance from the very beginning of my research work. His time to time suggestions have encouraged me to work hard in my research. His help has been instrumental from the beginning. All the work of this thesis is done with his consent. This work would not have been completed without his support and enthusiasm. He treated me like his own student through his valuable suggestions and vivid discussions. I am privileged to have the opportunity to work with him.

I would like to extend my appreciation to my SRC (Student Research Committee) members Prof. R. K. Sharma, Prof. N. Shravan Kumar and Prof. S. D. Joshi for their valuable time and suggestions. I am greatly indebted to all faculty members of Department of Mathematics IIT Delhi, for their cooperation and support. Also I would like to thank UGC (University Grants Commission) for providing me the financial assistance during the period of my research work.

I appreciate all my seniors for making me comfortable in the department and motivating me in the initial days of my research. It appears almost impossible to thank each one of my friends here

individually, but I would like them to know that the happy memories of my friendship with them always remain in the deepest corner of my heart. I am highly thankful to my colleagues and friends, Pawan, Anubha Jindal, Arti Pandey, Rachana Gupta, Lipsy, Pooja, for encouraging me and creating joyful environment in this duration.

Words cannot completely express my love and gratitude to my family who have supported and encouraged me through this journey. I would like to thank my parents for their life long support which sustained my interest in research and motivated me towards the successful completion of this study. I would also like to thank my husband Rohit Gupta and my son Rohansh Gupta for their love and support.

Last but not the least, I would like to thank the Almighty God for being with me and answering all my prayers.

New Delhi

June 2019

Ankita Jindal

Abstract

This thesis is devoted to the study of algebraic properties of Generalized Laguerre Polynomials (GLP) $L_n^{(\alpha)}(x) = (-1)^n \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}$, where α is an arbitrary real number, $n \in \mathbb{N}$ and the binomial coefficient $\binom{t}{k}$ is defined to be $\binom{t}{k} := \frac{t(t-1)\cdots(t-k+1)}{k!}$ for non-negative integer k . For different values of the parameter α , this family provides polynomials which are of great interest. In this thesis, we concern ourselves with two types of negative integral values of the parameter α .

We prove the irreducibility of the polynomial $L_n^{(-2n-\beta)}(x)$ for each $n \in \mathbb{N}$ and $1 \leq \beta \leq 4$ unless $n \equiv 0 \pmod{4}$ and $\beta = 4$, in which case it is either irreducible or a linear polynomial times an irreducible polynomial of degree $n - 1$. The case $\beta = 1$ corresponds to the Bessel polynomials which have been studied by Grosswald, Filaseta and Trifonov. Our ideas give a simpler proof of their results also.

We, furthermore, study the irreducibility and the Galois group of the polynomial $L_n^{(-1-n-r)}(x) = \sum_{j=0}^n \binom{n-j+r}{n-j} \frac{x^j}{j!}$ for each pair of integers $r \geq 0, n \geq 1$. Hajir conjectured that for integers $r \geq 0$ and $n \geq 1$, $L_n^{(-1-n-r)}(x)$ is an irreducible polynomial whose Galois group contains A_n , the alternating group on n symbols. Extending earlier results of Schur, Hajir, Sell, Nair and Shorey, we confirm this conjecture for all $r \leq 60$.

We also prove that $L_n^{(-1-n-r)}(x)$ is an irreducible polynomial whose Galois group contains A_n whenever $n > e^{r(1+\frac{1.2762}{\log r})}$.

सार

यह शोध सामान्यीकृत लैगुएरे बहुपदों (GLP) $L_n^{(\alpha)}(x) = (-1)^n \sum_{j=0}^n \binom{n+\alpha}{n-j} \frac{x^j}{j!}$ के बीजीय गुणों के अध्ययन के लिए समर्पित है, जहां α एक मनमानी वास्तविक संख्या है, $n \in \mathbb{N}$ और गैर-नकारात्मक पूर्णांक k के लिए, द्विपद गुणांक $\binom{t}{k}$ को $\binom{t}{k} := \frac{t(t-1)\dots(t-k+1)}{k!}$ परिभाषित किया गया है। पैरामीटर α के विभिन्न मूल्यों के लिए, यह परिवार ऐसे बहुपदों प्रदान करता है जो की बहुत रूचि वाले हैं। इस शोध में, हम अपने आप को पैरामीटर α के दो प्रकार के नकारात्मक अभिन्न मूल्यों के साथ सम्बद्ध करते हैं।

हम प्रत्येक $n \in \mathbb{N}$ और $1 \leq \beta \leq 4$ के लिए, बहुपद $L_n^{(-2n-\beta)}(x)$ की अटूटता साबित करते हैं जब तक $n \equiv 0 \pmod{4}$ और $\beta = 4$ न हो, जिस स्थिति में यह या तो अटूट है या डिग्री $n - 1$ के अटूट बहुपद के साथ एक रेखीय बहुपद का गुणक है। स्थिति $\beta = 1$, Bessel बहुपद से संबंधित है जिसका अध्ययन Grosswald, Filaseta and Trifonov द्वारा किया गया है। हमारे विचार उनके परिणामों का एक सरल प्रमाण भी देते हैं।

हम, इसके अलावा, पूर्णाकों की प्रत्येक जोड़ी $r \geq 0, n \geq 1$ के लिए, बहुपद $L^{-1-n-r}(x) = \sum_{j=0}^n \binom{n-j+r}{n-j} \frac{x^j}{j!}$ की अटूटता और Galois समूह का अध्ययन करते हैं। Hajir ने अनुमान लगाया कि पूर्णाकों $r \geq 0$ और $n \geq 1$ के लिए, $L^{-1-n-r}(x)$ एक अटूट बहुपद है जिसके Galois समूह में n प्रतीकों पर प्रत्यावर्ती समूह A_n शामिल है। Schur, Hajir, Sell, Nair और Shorey के पहले के परिणामों का विस्तार करते हुए, हम सभी $r \leq 60$ के लिए इस अनुमान की पुष्टि करते हैं।

हम ये भी प्रमाणित करते हैं कि जब भी $n > e^{r(1+\frac{1.2762}{\log r})}$ है $L^{-1-n-r}(x)$ एक अटूट बहुपद है जिसके Galois समूह में A_n शामिल है।

Contents

Certificate	i
Acknowledgements	iii
Abstract	v
List of Symbols	ix
Introduction	1
1 Preliminaries	7
1.1 Generalized Laguerre Polynomials	7
1.2 Newton polygons	8
1.3 Estimates of some functions on primes	11
1.4 An argument of Erdős	13
2 Irreducibility of the polynomials $L_n^{(-2n-\beta)}(x)$ and their Galois groups	15
2.1 Introduction	15
2.2 Notation and Preliminaries	20
2.3 Lemmas for $L_n^{(-2n-\beta)}(x)$ arising from Newton polygons	24

2.4	Proof of irreducibility of $L_n^{(-2n-\beta)}(x) : 1 \leq \beta \leq 4$	36
3	Irreducibility of the polynomials $L_n^{(-1-n-r)}(x)$	57
3.1	Introduction	57
3.2	Lemmas for the proof of Theorem 3.1.5 and Theorem 3.1.8	60
3.3	Proof of Theorem 3.1.5	66
3.4	Proof of Theorem 3.1.8	69
4	Galois groups of the polynomials $L_n^{(-1-n-r)}(x)$	71
4.1	Introduction	71
4.2	Lemmas for the proofs of Theorem 4.1.5 and Theorem 4.1.8	74
4.3	Proof of Theorem 4.1.5	77
4.4	Proof of Theorem 4.1.8	84
5	Conclusion and future work	85
5.1	Conclusion	85
5.2	Future Work	86
	Bibliography	89
	Bio-Data	93

List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
p	a prime
$a b$	a divides b
$a b$	$a b$ and $a^2 \nmid b$
$a \nmid b$	a does not divide b
$a \equiv b \pmod{n}$	a is congruent to b modulo n
$a \not\equiv b \pmod{n}$	a is not congruent to b modulo n
$\min(a, b)$	minimum of a and b
$\max(a, b)$	maximum of a and b
$\gcd(a, b)$	the greatest common divisor of a and b
$\forall$	for all
∞	infinity
$\emptyset$	the empty set
$ S $	the cardinality of the finite set S
$x \in X$	x is a member of X
$x \notin X$	x is not a member of X

S_n	the symmetric group on n symbols
A_n	the alternating group on n symbols
$\log x$	the logarithm of x to the base e