

**LAYER RESOLVING NUMERICAL METHODS  
FOR SOME SINGULARLY PERTURBED  
CONVECTION-REACTION-DIFFUSION  
INTERFACE PROBLEMS**

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# **Layer Resolving Numerical Methods for Some Singularly Perturbed Convection–Reaction–Diffusion Interface Problems**

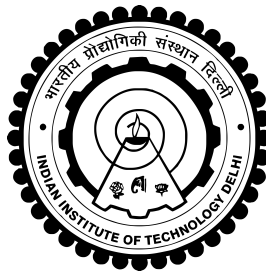
by

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*Submitted*

*in fulfillment of the requirements of the degree of Doctor of Philosophy  
to the*



**Indian Institute of Technology Delhi  
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**To  
My Grandmother**

# Certificate

This is to certify that the thesis entitled **Layer Resolving Numerical Methods for Some Singularly Perturbed Convection–Reaction–Diffusion Interface Problems** submitted by **Mr. Abhay Kumar Chaturvedi** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi  
October 2021

**Prof. S. Chandra Sekhara Rao**  
**Department of Mathematics**  
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# Abstract

Interface problems have always been the most challenging research area because of the low global regularity of the solution caused by the discontinuity and the high curvature of the interface exhibited by the interface's shape. The classical analysis of singularly perturbed interface problems is not yet developed and challenging to implement. The analysis and numerical solution of interface problems deservedly attract substantial attention. This thesis design and analyze numerical techniques for some singularly perturbed interface problems.

A two-dimensional singularly perturbed convection-reaction-diffusion problem that has discontinuities, along lines parallel to x- and y-axes, in the source term as well as in the convection and reaction coefficients is considered. The coefficient of the highest-order term is a small positive parameter. We propose a decomposition of the solution that yields sharp bounds on its derivatives. A particular finite difference scheme is constructed on an appropriate Shishkin mesh, and it is established that the computed solution is almost first-order parameter-uniformly convergent. For the same problem, a local discontinuous Galerkin (LDG) method is constructed on an appropriate Shishkin mesh. The test functions in the LDG method are piecewise polynomials that lie in the space  $\mathcal{Q}_r$  of piecewise polynomials of degree at most  $r$  in each variable, where  $r$  is a positive integer. We established that the error in the computed solution converges at the rate of  $r + \frac{1}{2}$  in a DG-norm.

We next design and analyze a parameters-uniform numerical method for a time-dependent weakly coupled system of two convection-diffusion equations that has a discontinuity, along the line  $x = d$ , in the source term. The second-order term of each equation is multiplied by a small singular perturbation parameter of different magnitude. The convergence analysis of the numerical method is given using the decomposition of the solution into regular and singular components. The numerical approximations produced by the numerical method are uniformly convergent of first-order in time and almost first-order in space with respect to both perturbation parameters.

A finite difference method for a system of  $m(\geq 2)$  singularly perturbed parabolic semilinear reaction-diffusion equations with a discontinuous source term having discontinuities along  $\Gamma_d = \{(x, t) : x = d, 0 < t \leq T\}$  is constructed. The semilinear operator is linearized, and a maximum principle is proved for the linearized operator. A Green's function is introduced for the semilinear operator using the linearized operator. Certain bounds on the Green's function and its derivatives are obtained. Using these bounds, we give computable parameter-uniform error estimates in maximum norm.

Next, we construct and analyze a numerical method for a time-dependent weakly coupled system of two singularly perturbed semilinear reaction-diffusion equations. The source terms in both equations have discontinuities in the spatial variables along  $x = d, d \in \Omega := (0, 1)$ . The highest order spatial derivatives in the first and second equations are multiplied by positive perturbation parameters  $\varepsilon_1$  and  $\varepsilon_2$ , respectively, which could be arbitrarily small. The domain is discretized using an appropriate Shishkin mesh. Using the decomposition of the solution, we obtain  $(\varepsilon_1, \varepsilon_2)$ -uniform error estimates in "maximum norm." We prove that the method is a  $(\varepsilon_1, \varepsilon_2)$ -uniformly convergent of first-order in time and almost second-order in space.

Numerical experiments are conducted on some test problems for all the developed numerical methods to validate the theoretical results.

# सार

अंतराफलक की समस्या हमेशा बहुत चुनौतीपूर्ण अनुसंधान क्षेत्र रही है इसका कारण असंतता और अंतराफलक के आकार द्वारा प्रदर्शित अंतराफलक की उच्च वक्रता के कारण समाधान की कम व्यापक नियमानुकूलता है। सिंगुलरली परटबर्ड अंतराफलक समस्याओं का पारम्परिक विश्लेषण अभी तक विकसित नहीं हुआ है और इसे कार्यान्वयन करना चुनौतीपूर्ण है। अंतराफलक समस्याओं का विश्लेषण और संख्यात्मक समाधान उचित रूप से पर्याप्त ध्यान आकर्षित करते हैं। यह शोध प्रबंध कुछ सिंगुलरली परटबर्ड अंतराफलक समस्याओं के लिए संख्यात्मक तकनीकों का निर्माण और विश्लेषण करती है।

एक द्वि-आयामी सिंगुलरली परटबर्ड संवहन-प्रतिक्रिया-प्रसार समस्या, जिसमें श्रोत पद के साथ-साथ संवहन और प्रतिक्रिया गुणांकों में  $x$ - और  $y$ -अक्षों के समानांतर रेखाओं के अनुदिश असंतता है, पर सुविचार किया गया है। अवकल समीकरण में उच्चतम-क्रम पद का गुणांक एक छोटा धनात्मक पैरामीटर है। हम समाधान के वियोजन का प्रस्ताव करते हैं जो इसके डेरिवेटिव पर ठीक-ठीक सीमा निश्चित करता है। एक विशेष परिमित अन्तराल तकनीकी उपयुक्त शिफ्टिंग मेस पर निर्माण की गई है, और यह स्थापित किया गया है कि अभिकलित समाधान लगभग प्रथम-क्रम पैरामीटर-यूनिफार्म रूप से अभिसरित होता है। उसी समस्या के लिए, एक उपयुक्त शिफ्टिंग मेस पर एक लोकल डिस्कंटीन्यूअस गैलर्किन (एलडीजी) विधि का निर्माण किया गया है। एलडीजी विधि में परीक्षण फलन कई खंडों में बहुपद हैं जो समष्टि  $Q_r$ , जो कि प्रत्येक चर में अधिकतम  $r$  घात के बहुपद हैं, में स्थित हैं, जहां  $r$  एक धनात्मक पूर्णांक है। हमने स्थापित किया है कि अभिकलित समाधान में त्रुटि  $r + \frac{1}{2}$  की दर से डीजी-मानदंड में अभिसरित होती है।

आगे हम समय पर निर्भर करने वाले दो संवहन-प्रसार समीकरणों की कमजोर युग्मित निकाय के लिए एक पैरामीटर-यूनिफार्म संख्यात्मक विधि का निर्माण और विश्लेषण करेंगे, जिसमें श्रोत पद में रेखा  $x = d$  के अनुदिस एक असंतता है। प्रत्येक समीकरण के दूसरे-क्रम के पद को अलग-अलग परिमाण के एक छोटे सिंगुलर परटबर्डसन पैरामीटर से गुणा किया जाता है। समाधान को रेगुलर और सिंगुलर घटकों में विभाजन का उपयोग करके संख्यात्मक विधि के अभिसरण का विश्लेषण दिया गया है। संख्यात्मक विधि द्वारा उत्पन्न संख्यात्मक अनुमान समय में प्रथम-क्रम से औरस्थानिक चर में लगभग प्रथम-क्रम से दोनों पैरामीटर के संदर्भ में एकसमान रूप से अभिसरित होता है।

$m(\geq 2)$  सिंगुलर्ली परटर्ब्ड पैराबोलिक सेमिलीनियर समीकरणों के एक निकाय, जिसके श्रोत पद में  $\Gamma_d = \{(x, t) : x = d, 0 < t \leq T\}$  के अनुदिस असंतता है, के लिए एक परिमित अन्तराल तकनीकी का निर्माण किया गया है। सेमिलीनियर ऑपरेटर को रैखिककृत किया गया है, और रैखिककृत ऑपरेटर के लिए एक अधिकतम सिद्धांत सिद्ध किया गया है। रैखिककृत ऑपरेटर का उपयोग करके सेमिलीनियर ऑपरेटर के लिए एक ग्रीन का फ़ंक्शन पेश किया गया है। ग्रीन के फ़ंक्शन और उसके डेरिवेटिव पर कुछ सीमाएं प्राप्त की गई हैं। इन सीमाओं का उपयोग करते हुए, हम अधिकतम-मानदंड में गणना योग्य पैरामीटर-यूनिफार्म त्रुटी का आकलन करते हैं।

आगे हम समय पर निर्भर करने वाली दो प्रतिक्रिया-प्रसार समीकरणों की कमजोर युग्मित निकाय के लिए एक संख्यात्मक विधि का निर्माण और विश्लेषण करते हैं। दोनों समीकरणों के श्रोत पदों में  $x = d, d \in \Omega := (0, 1)$ , के अनुदिस स्थानिक चर में असंतता है। पहले और दूसरे समीकरणों में उच्चतम क्रम के स्थानिक डेरिवेटिव्स को क्रमशः धनात्मक परटर्बेसन पैरामीटर  $\varepsilon_1$  और  $\varepsilon_2$  से गुणा किया गया है, जो मनमाने ढंग से छोटा हो सकता है। उपयुक्त शिफ्टिंग मेश का उपयोग करके डोमेन को विभाजीत किया गया है। समाधान के अपघटन का उपयोग करके, हम "अधिकतम मानदंड" में  $(\varepsilon_1, \varepsilon_2)$ -यूनिफार्म त्रुटी का अनुमान प्राप्त करते हैं। हम सिद्ध करते हैं कि विधि समय में पहले-क्रम और स्थानिक चर में लगभग दुसरे-क्रम से  $(\varepsilon_1, \varepsilon_2)$ -यूनिफार्म रूप से अभिसरित होती है।

सैद्धांतिक परिणामों की अभिपुष्टि करने के लिए सभी विकसित संख्यात्मक विधियों के लिए कुछ परीक्षण समस्याओं पर संख्यात्मक प्रयोग किए गए हैं।

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# List of Symbols

Symbol	Meaning
$i, j, k$	non-negative integers
$\Omega$	given space variable(s) domain
$\partial\Omega = \Gamma$	boundary of $\Omega$
$\Omega_1$ and $\Omega_2$	subdomains of $\Omega$
$G = \Omega \times (0, T]$	given domains for non-stationary problems
$G_1$ and $G_2$	subdomains of $G$
$L^p(\Omega)$	Lebesgue spaces, $1 \leq p \leq \infty$
$W^{k,p}(\Omega)$	Sobolev spaces, $1 \leq p \leq \infty$
$H^k(\Omega), H_0^k(\Omega)$	Sobolev spaces, $W^{1,2}(\Omega)$
$C^k(\Omega)$	function spaces
$C^{k,\theta}(\Omega)$	Hölder spaces with exponent $\theta \in (0, 1)$
$\varepsilon$	perturbation parameter
$\mathbf{T}$	semilinear differential operator
$\mathbf{a}(\cdot, \cdot)$	reaction function
$N$ and $M$	mesh parameters
$\mathbf{T}^{N,M}$	semilinear discrete operator
$\mathcal{G}$	Green's function
$C$	generic positive constant, independent of perturbation and mesh parameter(s)
$\mathbf{C}$	generic positive constant vector
$\ \phi\ _S$	maximum norm of $\phi$
$\mathbf{v} \leq (\geq) \mathbf{w}$	means $v_i \leq (\geq) w_i, \mathbf{v} = (v_1, \dots, v_m)^T, \mathbf{w} = (w_1, \dots, w_m)^T$

$ \mathbf{v}  = ( v_1 , \dots,  v_m )^T$	$\mathbf{v} = (v_1, \dots, v_m)^T$
$\nabla$	gradient operator
$\Delta$	Laplacian operator
$[[\cdot]]$	jump operator
$\{\{\cdot\}\}$	averaging operator
$U(x_i, y_j)$	discrete solution at node point $(x_i, y_j)$