ALGEBRAIC NONDETERMINISM AND TRANSITION SYSTEMS

by

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Submitted
in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

INDIAN INSTITUTE OF TECHNOLOGY, DELHI

February 2005

fuzzy sets

CERTIFICATE

This is to certify that the thesis entitled ALGEBRAIC NONDETERMINISM AND TRANSITION SYSTEMS which is being submitted by PRITI SINHA for the award of the degree of DOCTOR OF PHILOSOPHY in MATHEMATICS DEPARTMENT to the INDIAN INSTITUTE OF TECHNOLOGY DELHI, is a bonafide research work done under our guidance and supervision.

The thesis has reached the standard, fulfilling the requirements of the regulations relating to the degree. The results obtained in the thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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Acknowledgments

I am deeply indebted to my guide and mentor Dr. Wagish Shukla. Working and being with him has been a tremendous learning experience. His wisdom, care and support have helped me come to terms with some of the toughest periods of my life — academically, professionally and personally.

I am also extremely indebted to my co-supervisor, Prof. Sanjiva Prasad for the guidance and encouragement he has given me. He has always compelled me to reach out far and think beyond my limits. His ideas form some of the basic building blocks of my work.

I would like to express my sincere thanks to Prof. A. K. Srivastava for proffering his time generously and for his valuable comments on my work.

I would like to thank Prof. Suresh Chandra and Prof. J. B. Srivastava for encouraging me and for giving me the right advice at the right time. I also thank Prof. B. Chandra, Head of the Department, Mathematics, for the continuous encouragement I have received from her.

I thank my very dear friends Bhavana and Rajeev for their love and support and for being there for me whenever I needed them.

I also thank my colleague and friend Vidyottama Vijay for her love and support. She was always there to help me, readily and happily.

This thesis is dedicated to my father Y.N.P Sinha. He trusted me and always encouraged me to become a strong individual, with very strong values. Whenever I became weak and vulnerable, he was always there for me, to give his unconditional love and support. No one can be as sweet, loving and caring as my mother Uma Sinha; she tolerated me every time I was temperamental and her self-giving nature makes me very proud of her. I feel blessed to have been raised in such a wonderful family, with three great siblings and amazing parents. I am grateful to my sisters Vandana, Rupam and Aparna, my brother Vagish and my brother-in-law Mr. A. K. Sinha for loving me unconditionally and giving me encouragement and moral support. I also thank my brothers Amrit and Ashutosh for standing by me, specially at the time of submission.

I am also blessed to have great in-laws, my father-in-law R. Paulraj and mother-in-law Daisy Paulraj. I am extremely grateful to them for all the moral support and encouragement they have given me. I also thank my brother-in-law Satish for constantly boosting my morale and standing by me whenever I needed support.

I couldn't have made it to this point without the love and support of my husband Rajesh Paul, who has always encouraged me to pursue my dream. Rajesh's attentiveness to my needs, his willingness to listen when things were difficult and his good-hearted companionship through everything were essential to my completion of this thesis, and I am so grateful to him for his unwavering love and support.

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February 2005

Abstract

In this thesis we examin the concept of algebraic nondeterminism, by which we mean the introduction of the Kleisli category via a monad (T, η, μ). This semantic edge to the old concept of a monad was provided by E. G. Manes in the late seventies. In the nineties E. Moggi read into monads basic constructions of computer science and the happy double semantics has led to vigorous developments. We concentrate on nondeterminism in sets, taking Zadeh fuzzy sets as the prototype and in machines, with transition systems as the basic model.

With this in mind, we construct two conceptual economy tools, the category *Qua* and the Lawvere metric spaces.

The category *Qua* is capable of modelling both fuzzy sets and transition systems. We show that the category of $\mathbb{L}$-fuzzy sets [Gog67], category $Rel\mathbb{L}$ [Bar96], the category $M_L(C)$ (for the case $C = \mathcal{S}et$) [dP89], the category $\mathcal{P}\mathcal{V}$ [Bla95] and the category of games, $Game_{\mathbb{K}}$ [LS91] are special cases of the category *Qua*. The category of transition systems and simulations [WN93] is also a special case of *Qua*.
We construct an algebraic theory in *Qua*.
We further construct a generalization of the category *Qua*. This particularizes to a category of "poset-valued sets" which is a generic technique for building models of linear logic [SdP04].

We read a transition system as "a generalized metric space". This forces us to

develop a theory of "Lawvere metric spaces" from the beginning, obtained by distilling the logic of the closed interval $[0, \infty]$ (or $[0, 1]$) into a structure we have named $\mathbb{Q}$.

In both *Qua* and $\mathbb{Q}$, only the commutative fuzzy conjunctions have been considered. We also provide a general framework for noncommutative fuzzy conjunctions.

We examin in some detail the "totally fuzzy sets", better known as "Ω-sets" where Ω is a complete Heyting algebra, and provide a comparison with our Lawvere metric spaces.

In addition, definitions and points of view on nondeterministic set theory and machine theory that we have come across have been examined and provided different semantic edges with generalizations. For example, we introduce the concept of a transition system as a certain monoid morphism and also a certain kind of F-model where F is an endofunctor on *Set*.

These attempts cover most of the known instances of "fuzzy (finite state) machines", "the coalgebraic approach to transition systems", "metric bisimulations" and others.

Thus we have provided unification paradigms in algebraic nondeterminism for sets and machines based mostly on algebraic theories with a view to cover the current practice in fuzzy logic, linear logic and transition systems.

We close with some problems stemming from this dissertation in particular or the philosophy used in general; these include an investigation of the 2-categorical structure of *Qua*, the membership issue in fuzzy set theory, noncommutative Lawvere metric spaces and some others.

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