

ALGEBRAIC NONDETERMINISM AND TRANSITION SYSTEMS

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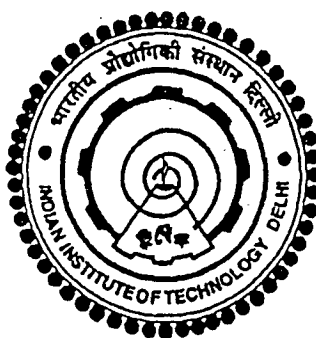
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DEPARTMENT OF MATHEMATICS

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

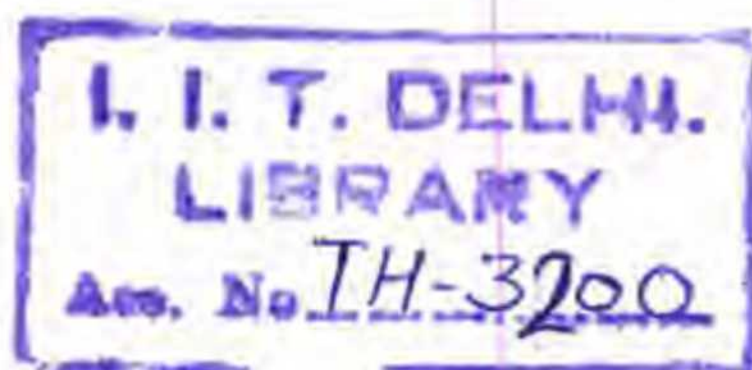
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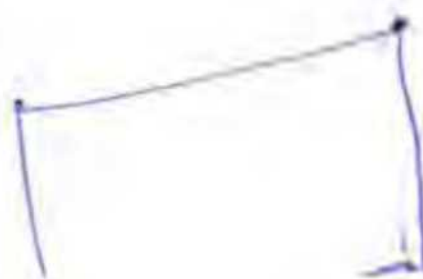
INDIAN INSTITUTE OF TECHNOLOGY, DELHI

February 2005

Survey sets



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CERTIFICATE

This is to certify that the thesis entitled ALGEBRAIC NONDETERMINISM AND TRANSITION SYSTEMS which is being submitted by PRITI SINHA for the award of the degree of DOCTOR OF PHILOSOPHY in MATHEMATICS DEPARTMENT to the INDIAN INSTITUTE OF TECHNOLOGY DELHI, is a bonafide research work done under our guidance and supervision.

The thesis has reached the standard, fulfilling the requirements of the regulations relating to the degree. The results obtained in the thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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Abstract

In this thesis we examine the concept of algebraic nondeterminism, by which we mean the introduction of the Kleisli category via a monad (T, η, μ) . This semantic edge to the old concept of a monad was provided by E. G. Manes in the late seventies. In the nineties E. Moggi read into monads basic constructions of computer science and the happy double semantics has led to vigorous developments. We concentrate on nondeterminism in sets, taking Zadeh fuzzy sets as the prototype and in machines, with transition systems as the basic model.

With this in mind, we construct two conceptual economy tools, the category *Qua* and the Lawvere metric spaces.

The category *Qua* is capable of modelling both fuzzy sets and transition systems. We show that the category of $\mathbb{L}$ -fuzzy sets [Gog67], category *Rel* $\mathbb{L}$ [Bar96], the category $M_L(C)$ (for the case $C = Set$) [dP89], the category $\mathcal{PV}$ [Bla95] and the category of games, *Game* $_{\mathbb{K}}$ [LS91] are special cases of the category *Qua*. The category of transition systems and simulations [WN93] is also a special case of *Qua*.

We construct an algebraic theory in *Qua*.

We further construct a generalization of the category *Qua*. This particularizes to a category of “poset-valued sets” which is a generic technique for building models of linear logic [SdP04].

We read a transition system as “a generalized metric space”. This forces us to

develop a theory of “Lawvere metric spaces” from the beginning, obtained by distilling the logic of the closed interval $[0, \infty]$ (or $[0, 1]$) into a structure we have named $\mathbb{Q}$.

In both Qua and $\mathbb{Q}$, only the commutative fuzzy conjunctions have been considered. We also provide a general framework for noncommutative fuzzy conjunctions.

We examine in some detail the “totally fuzzy sets”, better known as “ Ω -sets” where Ω is a complete Heyting algebra, and provide a comparison with our Lawvere metric spaces.

In addition, definitions and points of view on nondeterministic set theory and machine theory that we have come across have been examined and provided different semantic edges with generalizations. For example, we introduce the concept of a transition system as a certain monoid morphism and also a certain kind of F -model where F is an endofunctor on Set .

These attempts cover most of the known instances of “fuzzy (finite state) machines”, “the coalgebraic approach to transition systems”, “metric bisimulations” and others.

Thus we have provided unification paradigms in algebraic nondeterminism for sets and machines based mostly on algebraic theories with a view to cover the current practice in fuzzy logic, linear logic and transition systems.

We close with some problems stemming from this dissertation in particular or the philosophy used in general; these include an investigation of the 2-categorical structure of Qua , the membership issue in fuzzy set theory, noncommutative Lawvere metric spaces and some others.

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