

**ROBUST NUMERICAL SCHEMES FOR HYPERBOLIC BALANCE
LAWS**

ASHA KUMARI MEENA



**DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
FEBRUARY 2018**

©Indian Institute of Technology Delhi (IITD), New Delhi, 2018

ROBUST NUMERICAL SCHEMES FOR HYPERBOLIC BALANCE LAWS

by

ASHA KUMARI MEENA

Department of Mathematics

Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



Indian Institute of Technology Delhi
February 2018

Dedicated to

My Parents (Smt. Poonam Meena & Sh. Lalchand Meena)

Certificate

This is to certify that the thesis entitled **Robust Numerical Schemes for Hyperbolic Balance Laws** submitted by **Ms. Asha Kumari Meena** to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

New Delhi

February 2018

Dr. Harish Kumar

Assistant Professor

Department of Mathematics

Indian Institute of Technology Delhi

New Delhi 110016

Acknowledgements

At the completion of my Ph.D. thesis, I thank almighty God for his blessings on me to carry the strength and patience to reach this goal. I would like to take this opportunity to express my sincere gratitude to everyone who guided, helped and supported me throughout this journey.

First of all, I would like to convey my deepest gratitude to my Ph.D. supervisor Prof. Harish Kumar for all his guidance and valuable suggestions during the whole period. I am very much grateful to him for giving me this opportunity to pursue my doctoral programme under his supervision. I am thankful to him to make himself available to clarify my doubts in spite of his hectic schedule and to go through each part of the thesis. His deep insight, understanding, and passion for research have been of great value to improve the thesis. His continuous encouragement, motivation, and support gave me the strength to accomplish the work. This thesis would not have been possible without his support and guidance.

I would like to appreciate special gratitude to Prof. Praveen Chandrashekar for all the precious suggestions and guidance. His valuable discussions about computational methods and their implementation have significantly contributed to this thesis. I am also thankful to him to give me the opportunity to visit TIFR-CAM for the collaborative work.

My exclusive thanks go to my SRC (Student Research Committee) members: Prof. S. C. S. Rao, Prof. K. Sreenadh, Prof. B. Srinivasan and Prof. H. K. Malik for their valuable suggestions and reviews during the seminars. I appreciate very much thanks to all the faculty and staff members of Department of Mathematics, IIT Delhi for all the support and cooperation.

I pay lots of thanks to IIT Delhi authorities and security personnel for providing all the necessary facilities and security. I am thankful to IIT Delhi and UGC for providing research fellowship.

I would like to express my heartfelt gratitude and honor to most respected persons of my life, without whom I can not imagine my life, my mother, Smt. Poonam Meena and my father, Sh. Lalchand Meena. I feel

very much fortunate and blessed for being a daughter of such a humble and kind personality of you, Maa. My father's passion and dedication for his work have always been a big inspiration for me. My parent's immense affection and belief in my potential give me the strength to work hard. I was able to pursue my doctoral studies because they always stand by my decisions. Although they have gone through very hard phases during my Ph.D., they still provided a comfortable environment for my study. Every achievement of my life is a consequence of their unconditional love, support, blessings, sacrifices, and patience.

My brothers, Akhilesh and Rajesh, have always been pillars of my life. Although we spent few years of childhood altogether due to study, the moments which we spend together are the most precious and happy moments of my life. The long chat and gossip we do, have been the best medicine to reduce my stress. My brother Akhilesh's never giving up attitude inspires me a lot. I am profoundly grateful to my brother Rajesh (alumni of IIT Delhi) for being an inspiration to study in such a prestigious institute. I would like to express my thanks and affection to both of you for being there for me in every situation.

My special thanks and honor goes to my uncle (elder brother of my father) Sh. Ramdayal Meena and aunt Smt. Bhagwanti Meena. Their blessings and honored feeling at my achievements have been of great value for me. I would like to appreciate lots of thanks and love to my cousin brothers Vijay and Sandeep and cousin sister Ekta for all the care and affection. The moments I spend with them give me the great happiness.

I would like to thank my colleagues and friends: Anupam Gautam, Dr. Saroj Rani, Dr. Abhishake Rastogi, Chhanda Sen, Aparna Sharma, Dr. Varsha Srivastav, Dr. Manisha Srivastav, Dr. Anju Gupta, Dr. Reetu Siwach, Dr. Seema Kuswaha and Tuhina Mukherjee for providing a joyful and comfortable environment for research. I am heartily thankful to Anupam for sharing her time and listen to me patiently, whenever I needed. I will always remember the enjoyable moments we spent together. I can not forget to acknowledge my friends Inshu Chauhan, Sheela Verma and Rupali Mishra.

Last but not least, I express my deepest gratitude to the memories of my grandparents.

New Delhi

2018

Asha Kumari Meena

Abstract

The systems of hyperbolic balance laws arise in several areas, especially in Computational Fluid Dynamics (CFD) and plasma flows. Often the flux functions are nonlinear, which results in the nonexistence of classical solutions. So, a weak formulation of the systems is considered. However, the set of weak solutions contain more than one solution. Therefore, an additional criterion in the form of entropy inequality is imposed. More recently, it has been shown that even the entropy solutions are non-unique. So, in general, there is no well-posed result for the systems of hyperbolic conservation laws. Due to lack of theory for the continuous problem, designing of stable numerical algorithms for the systems of hyperbolic balance laws is highly non-trivial. In particular, two stability criteria are studied for a numerical method for hyperbolic PDEs, namely, positivity preservation (domain invariant) and entropy stability.

In this thesis, we design higher-order numerical schemes for systems of hyperbolic balance laws, which are positivity preserving. This is indeed desirable. Otherwise, we might lose the hyperbolicity of the system, and the solution produced by the numerical scheme will have no physical meaning. In particular, we aim to design positivity preserving finite volume and Discontinuous Galerkin (DG) methods for hyperbolic balance laws arising in some fluid and plasma flows.

First, we consider the Ten-Moment Gaussian closure hyperbolic system in two dimensions, which is used to model fluid flow. In contrast to Euler equations, the derivation of these equations does not assume *local thermodynamic equilibrium*, which allows anisotropic effects in the flow. This results in a tensorial description of pressure. So, to ensure positivity, we need to preserve positivity of symmetric pressure tensor. We design

a second order MUSCL type finite volume scheme which ensures this. This is achieved by combining a positivity preserving HLLC flux with appropriate linear reconstruction procedure. Furthermore, we extend these results to a Ten-Moment based model with source terms. The model is used in plasma flows to model matter laser interaction. The resulting numerical scheme is shown to more robust than the standard algorithms.

The Ten-Moment model with source terms also produces nontrivial stationary solutions (hydrostatic solutions) at steady state. A standard finite volume scheme does not preserve the hydrostatic solutions. So, we design a well-balanced second order finite volume scheme for the model. The scheme is well-balanced for general equation of state, provided the hydrostatic solution can be written as a function of space variables. The well-balanced property is achieved by a combination of hydrostatic reconstruction with contact preserving numerical flux and appropriate source discretization. The resulting scheme is shown to be far more accurate for the solutions close to steady states.

For the plasma flow model, we consider a Two-Fluid plasma flow balance law, where Ten-Moment equations model fluid components. The positivity preservation of density and 3×3 symmetric pressure tensor are key difficulties. We develop a positivity-preserving second-order finite volume scheme for the model. This is achieved by designing an HLLC solver for three-dimensional Ten-Moment equations, which is proven to be positivity preserving. To achieve second-order accuracy, we derive appropriate conditions on the slope of reconstruction. Furthermore, source terms are treated using both explicit and implicit discretizations.

In the second part of the thesis, we consider discontinuous Galerkin methods. First, we construct a positivity-preserving, arbitrary high-order discontinuous Galerkin scheme for Ten-Moment Gaussian closure equations with source term. The positivity of density and pressure tensor is achieved by designing a positivity limiters. This results in solving a cubic polynomial. The scheme is then shown to be more robust for the solutions close to vacuum states.

In the last part of this thesis, we consider the Two-Fluid plasma flow equations. Here, the flow of ions and electrons is modeled by Euler equations. This is then coupled with Maxwell's equations for electromagnetic quantities via stiff source term. We construct a second-order positivity-preserving nodal discontinuous Galerkin scheme for this model.

The positivity of Two-Fluid equations without source term is achieved by the positivity limiters. The source is then discretized implicitly and shown to be positivity preserving unconditionally. Both flux and source discretization are combined by Strang splitting method.

हाइपरबॉलिक बैलेन्स लॉज कई क्षेत्रों में उत्पन्न होते हैं, खासकर कम्प्यूटेशनल फ्लुइड डायनेमिक्स (सीएफडी) और प्लाज्मा फ्लो में। अक्सर फ्लक्स फंक्शंस नॉन-लीनियर होते हैं, जिससे क्लासिकल समाधानों की मौजूदगी नहीं होती है। इसलिए, सिस्टम्स के वीक फॉर्मूलेसन कन्सीडर किये जाते हैं। हालांकि, वीक समाधानों के सेट में एक से अधिक समाधान होते हैं। इसलिए, एंटरोपी इनइक्वलिटी के रूप में एक अतिरिक्त मानदंड लगाया जाता है। हाल ही में, यह दिखाया गया है कि एन्ट्रापी समाधान भी अनन्य हैं। अतः, सामान्य तौर पर, हाइपरबॉलिक कन्जरवेसन लॉज सिस्टम्स के लिए कोई वैल-पोस्ट परिणाम नहीं है। कॉन्टीनियस प्रोबलम्स के लिए थ्योरी की कमी के कारण, हाइपरबॉलिक बैलेन्स लॉज सिस्टम्स के लिए स्टेबल न्यूमेरिकल एल्गोरिद्म को डिजाइन करना बेहद नॉन-ट्रिवीअल है। विशेष रूप से, हाइपरबॉलिक पीडीई के लिए एक न्यूमेरिकल विधि के लिए दो स्टेबिलिटी मानदंडों का अध्ययन किया जाता है, अर्थात्, पाजिटिविटी-प्रिजर्वेशन (डोमेन इन्वारीअन्ट) और एन्ट्रापी स्टेबिलिटी।

इस थीसिस में, हम हाइपरबॉलिक बैलेन्स लॉज सिस्टम्स के लिए हाई-ऑर्डर न्यूमेरिकल स्कीम्स तैयार करते हैं, जो पाजिटिविटी प्रिजर्विंग हैं। यह वास्तव में वांछनीय है अन्यथा, हम सिस्टम की हाइपरबॉलिसिटी खो सकते हैं, और न्यूमेरिकल स्कीम द्वारा उत्पादित समाधान का कोई भौतिक अर्थ नहीं होगा। विशेष रूप से, हमारा लक्ष्य कुछ फ्लुइड और प्लाज्मा प्रवाह में उत्पन्न होने वाले हाइपरबॉलिक बैलेन्स लॉज के लिए पाजिटिविटी प्रिजर्विंग फिनिट वॉल्यूम और डिस्कन्टीन्यूस गलरकिन (डीजी) मेथड्स तैयार करना है।

सबसे पहले, हम टेन-मोमेंट गाऊसीयन क्लोसर हाइपरबॉलिक सिस्टम को दो डाइमेंशन्स में देखते हैं, जिसका उपयोग फ्लुइड फ्लो के मॉडल के लिए किया जाता है। यूलर एक्वेशन्स के विपरीत, इन एक्वेशन्स का व्युत्पन्न लॉकल थर्मोडायनामिक एक्विलिब्रियम नहीं मानता, जो फ्लो में अनिसोट्रॉपिक प्रभाव की अनुमति देता है। जिससे प्रेशर का टेन्सोरियल वर्णन होता है। अतः, पाजिटिविटी सुनिश्चित करने के लिए, हमें सिमेट्रिक प्रेशर टेन्सर की पाजिटिविटी बनाए रखने की आवश्यकता है। हम एक 2-ऑर्डर मुस्कल टाइप फिनिट वॉल्यूम स्कीम तैयार करते हैं जो यह सुनिश्चित करता है। यह उचित लीनियर रिक्स्ट्रक्शन प्रक्रिया के साथ पाजिटिविटी प्रिजर्विंग एचएलएलसी फ्लक्स को जोड़कर हासिल की है। इसके अलावा, हम इन परिणामों को टेन-मोमेंट बेस्ड मॉडल विथ सोर्स टर्म्स तक बढ़ाते हैं। यह मॉडल प्लाज्मा फ्लो में मेटर लेज़र इंटरैक्शन में उपयोग किया जाता है। परिणामी न्यूमेरिकल स्कीम को स्टैण्डर्ड एल्गोरिद्म से अधिक रोबस्ट दिखाया गया है।

टेन-मोमेंट बेस्ड मॉडल विथ सोर्स टर्म्स स्थिर स्टेट पर नॉनट्रिवैअल स्थिर समाधान (हीडरास्टाटिक समाधान) पैदा करता है। एक स्टैण्डर्ड फिनिट वॉल्यूम स्कीम हाइड्रोस्टेटिक समाधानों को प्रिजर्वे नहीं करती है। इसलिए, हम मॉडल के लिए एक वेल-बैलेंस्ड 2-ऑर्डर फिनिट वॉल्यूम स्कीम तैयार करते हैं। यह स्कीम सामान्य इक्वेसन ऑफ स्टेट के लिए वेल-बैलेंस्ड है, बशर्ते हाइड्रोस्टेटिक समाधान स्पेस वेरिबल्स के फंक्शन के रूप में लिखा जा सकता है। वेल-बैलेंस्ड प्रॉपर्टी, हाइड्रोस्टेटिक रिक्स्ट्रक्शन के साथ कांटेक्ट प्रिजर्विंग न्यूमेरिकल फ्लक्स और उचित सोर्स डिस्क्रिटाइजेसन के संयोजन द्वारा प्राप्त की जाती है। परिणामस्वरूप स्कीम स्थिर स्टेट के करीब के समाधान के लिए अधिक एक्यूरेट साबित हुई है।

प्लाज्मा फ्लो मॉडल के लिए, हम टू-फ्लुइड प्लाज्मा फ्लो बैलेंस लॉज पर विचार करते हैं, जहां टेन-मोमेंट एक्वेशन्स फ्लुइड कंपोनेंट्स को मॉडल करती है। डेंसिटी और 3x3 सिमेट्रिक प्रेशर टेन्सर का पाजिटिविटी प्रिजर्वेशन प्रमुख कठिनाइयां हैं। हम मॉडल के लिए एक पाजिटिविटी प्रिजर्विंग 2-ऑर्डर फिनिट वॉल्यूम स्कीम का विकास करते हैं। यह 3-डाइमेंशन्स टेन-मोमेंट एक्वेशन्स के लिए एक एचएलएलसी सॉल्वर को डिजाइन करके हासिल किया जाता है, जो कि पाजिटिविटी बनाए रखने के लिए सिद्ध हो रहा है। सेकंड-ऑर्डर एक्यूरेसी को प्राप्त करने के लिए, हम रिक्स्ट्रक्शन के स्लोप पर उचित कंडीशंस बनाते हैं। इसके अलावा, सोर्स टर्म्स के लिए दोनों एक्सप्लिसिट और इम्प्लिसिट डिस्क्रिटाइजेसन का उपयोग किया जाता है।

थीसिस के दूसरे हिस्से में, हम डिस्कन्टीन्यूस गलरकिन मेथड्स पर विचार करते हैं। सबसे पहले, हम टेन-मोमेंट मॉडल विथ सोर्स टर्म्स के लिए एक पाजिटिविटी-प्रिजर्विंग, आरबिटरेरी हाई-ऑर्डर डिस्कन्टीन्यूस गलरकिन स्कीम का निर्माण करते हैं। डेंसिटी और सिमेट्रिक प्रेशर टेन्सर की पाजिटिविटी एक पाजिटिविटी लीमीटर्स डिजाइनिंग के द्वारा प्राप्त की है। यह एक क्यूबिक पोलीनोमियल को सुलझाने में परिणाम है। इस योजना को वैक्यूम अवस्था के करीब के समाधान के लिए अधिक रोबस्ट दिखाया जाता है।

इस थीसिस के आखिरी भाग में, हम टू-फ्लुइड प्लाज्मा फ्लो एक्वेशन्स पर विचार करते हैं। यहां, आयनों और

इलेक्ट्रॉनों का प्रवाह यूलर एक्वेशन्स द्वारा किया जाता है। यह तब स्टिफ सोर्स टर्म्स के माध्यम से विद्युत चुम्बकीय क्वान्टिटीएस के लिए मैक्सवेल एक्वेशन्स के साथ युग्मित है। हम इस मॉडल के लिए सेकंड-ऑर्डर पाजिटिविटी-प्रेसेर्विंग नोडल डिस्कन्टीन्यूस गलरकिन स्कीम का निर्माण करते हैं। सोर्स टर्म्स के बिना टू-फ्लूइड एक्वेशन्स की पाजिटिविटी पाजिटिविटी लीमीटर्स से प्राप्त होती है। सोर्स इम्प्लीसिटली डिस्क्रटाइज की जाती है और बिना शर्त के पाजिटिविटी-प्रेसेर्विंग दिखाया जाता है। दोनों फ्लक्स और सोर्स डिस्क्रटाइजेसन स्ट्रॉंग स्प्लिटिंग विधि द्वारा संयुक्त किये जाते हैं।

Contents

Certificate	i
Acknowledgements	iii
Abstract	v
List of Figures	xiii
List of Tables	xix
List of Symbols	xxi
Introduction	1
1 Introduction to Hyperbolic Balance Laws	9
1.1 Weak Solutions	11
1.2 Entropy Solution	12
1.3 Invariant Domain	13
1.4 Riemann Problem	14
1.5 Examples	15
2 Finite Volume Schemes for Hyperbolic Balance Laws	19
2.1 Finite Volume Scheme	20
2.1.1 Examples	21
2.2 MUSCL Scheme	24

2.3	High Order Finite-Volume Schemes	25
2.4	Time Discretization	27
3	Robust MUSCL Schemes for Ten-Moment Equations with Source Terms	29
3.1	Ten-Moment Equations with Source Term	30
3.2	First Order Schemes for Homogeneous Model	37
3.2.1	HLLC Numerical Flux for Ten-Moment Equations	38
3.3	Second Order Schemes for Homogeneous Model	41
3.3.1	General Slope limiting of Primitive Variable (GSPV)	43
3.3.2	Conservative Slope limiting of Primitive Variable (CSPV)	46
3.4	Discretization of the Source Terms	49
3.4.1	Explicit Euler Source Update	50
3.4.2	Implicit Euler Source Update	51
3.5	Time Discretization	52
3.6	Numerical Results	53
3.6.1	Numerical Results for Homogeneous Case	53
3.6.2	Numerical Results: Non-Homogeneous Case	62
4	Well-balanced Schemes for Ten-Moment Equations with Source Term	69
4.1	Hydrostatic Solution	70
4.1.1	Hydrostatic Solutions in One-dimension	70
4.1.2	Hydrostatic Solutions in Two-dimensions	72
4.2	Well-Balanced Numerical Scheme	75
4.2.1	One-dimensional Finite Volume Schemes	76
4.2.2	Two-dimensional Finite Volume Schemes	81
4.3	Numerical Results	85
4.3.1	One-dimensional Numerical Tests	85
4.3.2	Two-dimensional Numerical Tests	98
5	Robust Schemes for Ten-Moment Two-Fluid Plasma Equations	109
5.1	Ten-Moment Two-Fluid Plasma Equations	110
5.2	Numerical Scheme for Homogeneous Model	113

5.2.1	First-order Scheme	114
5.2.2	Second-order Scheme	123
5.3	Discretization of Source Term	137
5.3.1	Explicit Discretization	138
5.3.2	Implicit Discretization	138
5.4	Time Discretization	140
5.5	Numerical Results	141
5.5.1	Accuracy Test	141
5.5.2	Soliton Propagation	143
5.5.3	Brio-Wu Shock Tube Test	144
5.5.4	Soliton Propagation in Two-dimensions	147
6	Robust Discontinuous Galerkin Schemes for Ten-Moment Equations	151
6.1	Discontinuous Galerkin Scheme	152
6.1.1	TVB Characteristic Limiter	153
6.1.2	Positivity Preserving Limiter	155
6.1.3	Source Update	163
6.2	Time Discretization	166
6.3	Numerical Results	167
6.3.1	Numerical Results for Ten-moment without Source	169
6.3.2	Numerical Results for Ten-moment with Source	176
7	Robust DG Schemes for Two-Fluid Plasma Equations	181
7.1	Two-Fluid Plasma Equations	182
7.2	Nodal Discontinuous Galerkin Scheme for Homogeneous Model	186
7.2.1	TVB Limiter	188
7.2.2	Positivity Preserving Limiter	189
7.3	Source Discretization	194
7.4	Numerical Results	196
7.4.1	Accuracy Test	196
7.4.2	Soliton Propagation	196
7.4.3	Brio-Wu Shock Tube Test	199

7.4.4 Soliton Propagation in Two-dimensions	203
8 Conclusion and Future Research	207
8.1 Contributions	207
8.2 Future Research	209
References	211
Curriculum Vitae	219

List of Figures

3.1	Convergence Rate: L1-error of density for CSPV, GSPV and MinMod solutions using HLLC solver. All the schemes achieve second order of accuracy.	54
3.2	Sod Shock Tube Problem: Numerical Solutions of MinMod, CSPV and GSPV limiters with HLLC flux using 100 cells at time $t = 0.125$. HLLC solver is used as numerical flux.	56
3.3	Two Shock Waves Problem: Numerical Solutions of MinMod, CSPV and GSPV limiters with HLLC flux using 100 cells. HLLC solver is used as numerical flux.	57
3.4	Two Rarefaction Waves Riemann Problem: Numerical Solutions of MinMod, CSPV and GSPV limiters with HLLC flux using 100 cells. HLLC solver is used as numerical flux.	58
3.5	Two Rarefaction Waves with Near Vacuum State: Numerical Solutions of CSPV and GSPV limiters with HLLC flux using 100 cells. HLLC solver is used as numerical flux.	59
3.6	Two Dimensional Near Vacuum State: Numerical Solutions of CSPV and GSPV limiters with HLLC flux using 100×100 cells. HLLC solver is used as numerical flux.	61
3.7	Two Rarefaction Waves with Gaussian Source Terms: Numerical Solutions of CSPV and GSPV limiters with HLLC flux using 500 cells. HLLC solver is used as numerical flux.	63

3.8	Two Dimensional Near Vacuum State: Numerical Solutions of CSPV and GSPV limiters with HLLC flux using 100×100 cells. HLLC solver is used as numerical flux.	64
3.9	Two Dimensional Realistic Simulation with Absorption Coefficient $v_T = 0$: Numerical Solutions of CSPV and GSPV limiters with HLLC flux using 100×100 , 200×200 and 400×400 cells along line $y = 50$	65
3.10	Two Dimensional Realistic Simulation with Absorption Coefficient $v_T = 1$: Numerical Solutions of CSPV and GSPV limiters with HLLC flux using 100×100 , 200×200 and 400×400 cells along line $y = 50$	66
3.11	Two Dimensional Realistic Simulation Comparison for Absorption Coefficient $v_T = 0$ and $v_T = 1$: Numerical Solutions of CSPV and GSPV limiters with HLLC flux using 400×400 cells.	67
3.12	Two Dimensional Realistic Simulation with Absorption Coefficient $v_T = 0$ and $v_T = 1$: Numerical Solutions of CSPV limiters with HLLC flux using 400×400 cells.	67
4.1	Order of convergence for density and pressure component p_{11}	86
4.2	Small Perturbation Test: Plots of perturbations in p_{11} using the second-order Well-balanced and non Well-balanced schemes with 100, 200, 400, 800 and 1000 resolutions.	91
4.3	Sod Shock Tube Test: Plots of numerical solutions for the second-order Well-balanced scheme using 100 and 2000 cells.	93
4.4	Contact Discontinuity Test: Plots of numerical solution of second-order, Well-balanced scheme and non Well-balanced schemes.	94
4.5	Discontinuous Potential Test: Plots of numerical solutions of second-order Well-balanced and non Well-balanced schemes.	95
4.6	Small Perturbation Test with Discontinuous Potential: Plots of perturbations in p_{11} using the second-order Well-balanced and non Well-balanced schemes.	97

4.7	Isothermal Small Perturbation Test in Two-dimension: Plots of perturbations in p_{11} using the second-order Well-balanced and non Well-balanced schemes using 50×50 cells.	101
4.8	Isothermal Small Perturbation Test in Two-dimension: Plots of perturbations in $\det(\mathbf{p})$ using the second-order Well-balanced and non Well-balanced schemes using 50×50 cells.	102
4.9	Discontinuous Potential in Two-dimensions: Numerical solution of the second-order Well-balanced and non Well-balanced schemes.	104
4.10	Small Perturbation Test with Discontinuous Potential in Two-dimension: Plots of perturbations in p_{11} using the second-order Well-balanced and non Well-balanced schemes using 50×50 cells.	105
4.11	Small Perturbation Test with Discontinuous Potential in Two-dimension: Plots of perturbations in $\det(\mathbf{p})$ using the second-order Well-balanced and non Well-balanced schemes using 50×50 cells.	106
5.1	Accuracy Test: Plots of order of convergence for ion density.	141
5.2	Soliton Propagation: Numerical evolution of solitons in ion density with 3000 cells and Larmor radius 10^{-2}	142
5.3	Soliton Propagation: Numerical evolution of solitons in ion density with 3000 cells and Larmor radius 10^{-4}	142
5.4	Soliton Propagation: Numerical evolution of solitons in ion density with 3000 cells and Larmor radius 10^{-6}	143
5.5	Brio-Wu Shock Tube Test: Numerical results for ion density with 10000 cells and Larmor radius 100.	145
5.6	Brio-Wu Shock Tube Test: Numerical results for ion density with 10000 cells and Larmor radius 1.	145
5.7	Brio-Wu Shock Tube Test: Numerical results for ion density with 10000 cells and Larmor radius 10^{-1}	146
5.8	Brio-Wu Shock Tube Test: Numerical results for ion density with HLLEM solve, 1000 cells and Larmor radius 10^{-3}	146

5.9	Soliton Propagation in Two-dimensions: Numerical evolution of solitons in ion density by O2-exp scheme with 100×100 cells and Larmor radius 10^{-2}	148
5.10	Soliton Propagation in Two-dimensions: Numerical evolution of solitons in ion density by O2-imp scheme with 100×100 cells and Larmor radius 10^{-2}	149
5.11	Soliton Propagation in Two-dimensions: One dimension cut of numerical evolution of solitons in ion density at $t = 0.3$ along $y = 1$ by O2-exp and O2-imp schemes with 100×100 cells and Larmor radius 10^{-2}	149
6.1	Convergence Rate: L1-error of density. The DG scheme achieve second-order accuracy for $k=1$ and third-order accuracy for $k=2$	167
6.2	Sod Shock Tube Problem: Numerical Solution for Second-order and Third-order RKDG method using 200 cells.	168
6.3	Two Shock Waves Problem: Numerical Solution of Second-order and Third-order RKDG method using 200 cells.	170
6.4	Two Rarefaction Waves Problem: Numerical Solution of Second-order and Third-order RKDG method using 200 cells.	171
6.5	Near Vacuum State Problem: Numerical Solution of Second-order and Third-order RKDG method with HLLC numerical flux using 100 cells with TVB parameter $M = 0$ and $M = 10^3$	173
6.6	Near Vacuum Test case in Two-dimension: Numerical solution (Density) of second-order and third-order RKDG methods using 100×100 cells.	174
6.7	Near Vacuum Test case in Two-dimension: One-dimensional cut of numerical solution of second-order and third-order RKDG methods using 100×100 cells along $y = 0$ line.	175
6.8	Two Rarefaction Waves with Gaussian Source : Numerical solution of Second-order and Third-order RKDG scheme using 500 cells.	175
6.9	Uniform State with Gaussian Source in Two-dimension: Numerical solution (Density) of second-order and third-order RKDG methods using 100×100 cells.	178

6.10	Uniform State with Gaussian Source in Two-dimension: One-dimension cut of the Numerical Solution (Density) for Second-order and Third-order RKDG methods using 100×100 cells along $y = -x + 4$ line.	179
6.11	Realistic Simulation with Absorption Coefficient $v_T = 0$ and $v_T = 1$: Two-dimensional Numerical Solution (Density) of Third-order RKDG methods using 400×400 cells.	179
6.12	Realistic Simulation with Absorption Coefficient $v_T = 0$ and $v_T = 1$: One-dimension cut of the numerical solution of third-order RKDG methods using 100×100 , 200×200 and 400×400 cells along $y = -x + 4$ line. . .	180
7.1	Accuracy Test: The L^1 -error plot of numerical solution for ion-density and second order reference slope.	197
7.2	Soliton Propagation: Numerical evolution of solitons in ion density by nodal DG scheme with 3000 cells and Larmor radius 10^{-2}	198
7.3	Soliton Propagation: Numerical evolution of solitons in ion density by nodal DG scheme with 3000 cells and Larmor radius 10^{-4}	198
7.4	Soliton Propagation: Numerical evolution of solitons in ion density by nodal DG scheme with 3000 cells and Larmor radius 10^{-6}	199
7.5	Brio-Wu Shock Tube Test: Numerical results for ion density by nodal DG scheme using HLLE Riemann solvers with 1024 cells and Larmor radius 100.200	
7.6	Brio-Wu Shock Tube Test: Numerical results for ion and electron number density by nodal DG scheme using HLLE Riemann solvers with 10016 cells and Larmor radius 10^{-1}	201
7.7	Brio-Wu Shock Tube Test: Numerical results for ion density and electron number density by nodal DG scheme using HLLE Riemann solvers with 1024 cells and Larmor radius 10^{-3}	202
7.8	Brio-Wu Shock Tube Test: Numerical results for ion density and electron number density by nodal DG scheme using HLLE Riemann solvers with 20000 cells and Larmor radius 10^{-3}	204

7.9	Soliton Propagation in Two-dimensions: Numerical evolution of solitons in ion density by nodal DG scheme with 100×100 cells and Larmor radius 10^{-2}	205
7.10	Soliton Propagation in Two-dimensions: Numerical evolution of solitons in ion density by nodal DG scheme with 100×100 cells and Larmor radius 10^{-4}	205
7.11	Soliton Propagation in Two-dimensions: One dimension cut of numerical evolution of solitons in ion density at $t = 0.3$ along $y = 1$ by nodal DG scheme with 100×100 cells and Larmor radius 10^{-2}	206
7.12	Soliton Propagation in Two-dimensions: One dimension cut of numerical evolution of solitons in ion density at $t = 0.3$ along $y = 1$ by nodal DG scheme with 100×100 cells and Larmor radius 10^{-4}	206

List of Tables

3.1	Initial Conditions for Sod’s Shock Tube Riemann Problem	55
3.2	Initial conditions for Two Shock Waves Riemann Problem for Homogenous Case	55
3.3	Initial conditions for Two Rarefaction Waves Riemann Problem	60
3.4	Initial conditions for Two Rarefaction Waves Riemann Problem: Near Vacuum Test Case	60
3.5	Initial conditions for Two Rarefaction Waves with Gaussian Source Terms	63
3.6	Simulation time for Two Rarefaction Waves with Gaussian Source Terms	63
3.7	Initial conditions for Uniform Plasma with Gaussian Source in Two Di- mension	64
3.8	Simulation time for Uniform Plasma State with Gaussian Source Terms .	65
4.1	Order of Accuracy: L^1 -errors in density and pressure component p_{11} . . .	86
4.2	Polytropic Well-balanced Test: L^1 -errors for the density, velocity and pres- sure components	87
4.3	Isentropic Well-balanced Test: L^1 -errors for the density, velocity and pres- sure components.	88
4.4	Isothermal Well-balanced Test: L^1 -errors for the density, velocity and pres- sure components.	89
4.5	Small Perturbation Test: Order of Accuracy using non Well-balanced Scheme: L^1 -errors in density.	90

4.6	Small Perturbation Test: Order of Accuracy using Well-balanced Scheme: L^1 -errors in density.	90
4.7	Initial conditions for Sod shock tube test	92
4.8	Initial conditions for Contact Discontinuity Test	94
4.9	Discontinuous Potential Test: L^1 -errors for density, velocity and pressure components.	95
4.10	Two-dimensional Polytropic Well-balanced Test: L^1 -errors for density, ve- locity and pressure components.	98
4.11	Two-dimensional Isentropic Well-balanced Test: L^1 -errors for density, ve- locity and pressure components.	99
4.12	Two-dimensional Isothermal Well-balanced Test: L^1 -errors for density, ve- locity and pressure components.	99
4.13	Discontinuous Potential Test in Two-dimensions: L^1 -errors for density, velocity and pressure components.	103
5.1	Accuracy Test: L^1 -error of ion density.	141
5.2	Computational time for Soliton Propagation Test case	144
5.3	Initial Conditions for Brio-Wu Shock Tube Test Case	147
6.1	Initial Conditions for Sod's Shock Tube Test Case	170
6.2	Initial conditions for Two Shock Waves Test Case	172
6.3	Initial conditions for Two Rarefaction Waves Test Case	172
6.4	Initial conditions for Near Vacuum Test Case	174
6.5	Initial conditions for two-rarefaction Riemann problem with Gaussian Source	177
7.1	Initial Conditions for Brio-Wu Shock Tube Test Case	199

List of Symbols

\mathbb{R}	Set of Real numbers
\mathbb{Z}	Set of Integers
\mathbb{N}	Set of Natural numbers
\mathbb{R}_+	Set of positive Real numbers
\mathbb{R}^d	Set of d-dimensional Real numbers
\subset	Subset
\in	Belongs to
\forall	For every
\implies	Implies
\otimes	Tensor outer product
Bold letters (<i>e.g.</i> \mathbf{u} , \mathbf{w})	Vector
∇	Gradient
∇	Divergence
$\nabla \times$	Curl
\mathbf{C}^1	Continuously differentiable function
\mathbf{C}^∞	Continuously differentiable function for all degrees
$\mathbf{C}_0^1(\mathbb{R}^d \times [0, \infty))$	\mathbf{C}^1 function with compact support in $\mathbb{R}^d \times [0, \infty)$
L_{loc}^∞	Space of locally bounded measurable functions
$(\cdot)^\top$	Transpose
$Sym(A)$	$\frac{1}{2}(A + A^\top)$

Introduction

Equations of the evolution of conserved quantities are the basis of several mathematical models, arising in physics, chemistry, engineering, and biology. They state that the rate of change of conserved quantities in any given domain depends on the flux through the boundary of the domain. The conservation laws balanced by source terms are known as balance laws. The systems of balance laws are characterized by eigenvalues and eigenvectors of the Jacobian of the flux function. A system of n equations of balance laws, having n real eigenvalues, and corresponding n linearly independent eigenvectors is called hyperbolic system. The models based on hyperbolic balance laws extensively arise in Computational Fluid Dynamics (CFD) and related areas, e.g., plasma flows. These represent the evolution of conserved quantities, which are governed by flux at the boundaries and source terms. Some of the examples are Shallow water equations, Euler equations of compressible flows and Magnetohydrodynamics (MHD) equations for plasma flows (See [2, 13, 24, 40]).

Often, the flux function is nonlinear. Due to this, the corresponding Cauchy problem does not exhibit classical solution, even for the given smooth data. This is due to the intersection of characteristic curves. To overcome this a *weak formulation* of the equations is considered, and one looks for the *weak solution*. The formulation is the generalization of the classical formulation in the sense that a classical solution is also a weak solution and any smooth weak solution can be shown to a classical solution ([25]). An equivalent criterion in the form of *Rankine-Huguniot* condition is derived for checking if the given function is a weak solution or not.

For a given Cauchy problem of the hyperbolic system, the weak solutions, in general,

are not unique. So, an additional criterion in the form of entropy condition is imposed to weed out solutions which are not physically relevant. The solutions which satisfying entropy condition are called entropy solutions. For scalar conservation laws, Kruzkov [37] proved the uniqueness of the entropy solutions. However, this is not the case when one considers the systems. Recently [14], it has been shown that for the system of Euler equations, we have nonunique entropy solutions. So, well-posedness is still an open problem for the hyperbolic systems.

The numerical schemes to approximate the conservation laws are often based on finite volume method and finite difference method, [24, 25, 61]. In these methods, spatial domain is discretized into a mesh of cells and cell averages are advanced by computing the numerical flux through the boundary. The numerical fluxes are based on exact or approximated Riemann solvers. The spatial high-order accuracy is achieved by reconstruction process, e.g., TVD (Total Variation Diminishing), ENO (Essentially Non-Oscillatory), WENO (Weighted ENO), HWENO (Hermite HWENO), [15, 40, 43, 44, 61]. The SSP (Strong Stability Preserving) Runge-Kutta methods are used to achieve high-order accuracy in time, [16, 18–20]. For a system of conservation laws with source terms, explicit and implicit discretization strategies are used for source terms and then combined with the discretization of flux by splitting methods, e.g., Godunov splitting, Strang splitting [61].

The Discontinuous Galerkin (DG) methods are another popular methods for discretization of conservation laws. They were first introduced for transport equations by Hill and Reed [32] and extended for hyperbolic PDEs by Cockburn and Shu, [17]. In DG methods, polynomial values are assigned to each cell, which are allowed to be discontinuous at the boundaries of cells. Limiters are applied to ensure non-oscillatory behavior of solutions. Similar to finite volume case, SSP Runge-Kutta methods are used to get high-order accuracy in time in DG schemes, which result in RKDG schemes. The DG methods can be seen as a generalization of finite volume methods and often results in significantly more accurate solution than the finite volume methods of comparable order. For scalar conservation law, the convergence results of the numerical methods are presented in [17, 25].

To ensure a consistent and stable discretization of the continuous models, we need the

discrete solution to have the properties of the continuous solution. However, due to lack of theoretical results, there are very few nonlinear stability estimates at the continuous level. An important stability feature of the continuous problem is that, given initial solution in a set of physically admissible state space, solution belongs to the same set for all time [8]. This property is also called *domain invariant* property. This is of utmost importance. Otherwise, we may lose the hyperbolicity of the system. For example for the Euler equations of compressible flows, this is equivalent of preserving the positivity of the density and the pressure.

The central theme of this thesis is to design of positivity preserving numerical methods for hyperbolic systems arising in CFD and plasma flows. For the first order finite volume schemes, this depends on the choice of numerical flux function. There are several approximated Riemann solvers based numerical fluxes which ensure the positivity of first-order schemes, e.g., Lax-Friedrich, HLLE, HLLC, etc. (See [61]). For the higher-order schemes, positivity depends on numerical flux and reconstruction procedure. The high order positivity preserving finite volume schemes for the Euler equations are studied in [6, 15, 33, 53, 69].

In the first contribution of this thesis, we consider a plasma flow model based on the two-dimensional Ten-Moment model equations ([7]). These equations are proposed by Levermore *et. al* (See [41, 42]). Unlike the Euler equations of compressible flow, here *local thermodynamic equilibrium* is not assumed. This is necessary for many applications (specially related to the plasma flows, see [5, 7, 10, 21, 28, 35, 50, 56, 57]). These equations describe the pressure as the symmetric tensor, and they are hyperbolic [7]. To ensure the positivity for this system, we need to preserve positivity of density and 2×2 symmetric pressure tensor. In [56], an HLLC based Riemann solver is presented for a Ten-Moment based plasma model and resulting first order scheme is shown to be positivity preserving. Also, a relaxation based scheme is proposed in [5], which ensures positivity and is entropy stable. However, the scheme is only first order. More recently, in [7], a Ten-Moment based plasma flow model with source terms is discretized to simulate laser effects. The discretization is based on an equivalent relaxed model, which take source terms also in consideration. The resulting first order scheme is shown to be positivity preserving and entropy stable. In this work, we propose positivity-preserving second order

discretization of the Ten-Moment equations with source terms. For the numerical flux, we use positivity preserving Riemann solvers. There are several examples of them, e.g., Lax-Friedrich, HLLE, and HLLC solvers (See [31, 56]). To obtain second-order positivity preserving scheme, we follow [4] and [62], and propose a robust MUSCL reconstruction process. We propose two slope limiters based on the reconstruction of the primitive variable, namely: Generalized slope limiter and Conservative slope limiters. We prescribe exact conditions for these schemes to be positivity preserving. The source is discretized using both explicit and implicit schemes. In the case of the explicit scheme, a condition on time step is derived, which ensures positivity of the solutions. The implicit treatment of the source is shown to be unconditionally positivity preserving. Furthermore, we show that we do not need to solve any system of equations to implement implicit source update. Both source and flux discretizations are then combined using Strang splitting, to ensure positivity of the whole scheme.

The plasma flow model based on the two-dimensional Ten-Moment model equations ([7]) has source terms which model laser-plasma interaction. The model exhibit non-trivial hydrostatic steady states. In such cases, it is desirable to have a well-balanced numerical method. For the systems of hyperbolic balance laws, there are several numerical schemes, which are designed to be well-balanced. In particular, for the case of Euler equations with gravitational source term, there are well-balanced schemes (See [11, 12, 36, 63, 64]). In [64], authors propose a well-balanced kinetic scheme. The scheme is shown to be well-balanced for the isothermal equation of state, which is achieved by extending the well-balanced scheme for shallow water equations from [54]. In [63], authors design a higher order, well-balanced, finite difference, WENO scheme. The scheme is again well-balanced for the isothermal case. For the isentropic case, a high order finite volume well-balanced scheme is designed in [36]. The scheme is shown to preserve the nontrivial stationary solutions by using hydrostatic reconstruction with consistent numerical flux and suitable source discretization. More recently, in [11], the author designed a second-order finite volume well-balanced scheme for both isothermal and polytropic solutions. In this work, we propose a well-balanced second order finite volume scheme for Ten-Moment equations with a potential source. This is perhaps the first well-balanced numerical scheme developed to approximate the Ten-Moment equations with a source. The work is

based on a generalization of the well-balanced numerical schemes proposed in [11, 23]. The key idea is to express steady states (polytropic, isentropic and isothermal) as a function of space variables. To achieve the well-balanced property, we first reconstruct ordered set of scaled variables, which is then combined with a contact preserving consistent numerical flux. This coupled with the appropriate discretization of the source results in a well-balanced scheme.

A plasma flow consists of ions, electrons, and neutral particles. These particles interact through both short range (e.g., collisions) and long range (e.g., electromagnetic) forces. They are increasingly used in spacecraft propulsion, controlled nuclear fusion and circuit breakers in the electrical power industry. Under the assumption of quasi-neutrality (i.e., charge density difference between ions and electrons is neglected), the flow of plasmas is modeled by the ideal MHD equations. Although the ideal MHD equations have been successfully employed in modeling and simulating plasma flows, this model is derived by ignoring the Hall effect and treating plasma flows as single fluid flows. These effects are significant for numerous applications, e.g., space plasmas, Hall current thrusters, field reversal configurations for magnetic plasma confinement and magnetic reconnection. In this work, we consider a more general two-fluid model. In the Two-Fluid equations, electrons and ions are treated as different fluids by allowing them to have different velocities and temperatures. Furthermore, if the assumption of *local thermodynamical equilibrium* not valid, we have to treat pressure using pressure tensor (See [28, 35]). This gives rise to the Ten-Moment Two-Fluid plasma equations, [28, 35]. It is a combination of Ten-Moment equations for ion and electron flow and Maxwell equations for the evolution of electromagnetic quantities. These equations are combined with a stiff source term which is based on the Lorentz force and rotation of pressure tensor by the magnetic field.

In this work, we aim to extend the robust scheme designed for the two-dimensional Ten-Moment model, to the three dimensional Two fluid, Ten-moment plasma model. In addition to the additional difficulty of preserving the positivity of pressure tensor in three dimensions, we also need to overcome the stiff source terms. We achieve this by designing an HLLC solver for three-dimensional Ten-Moment equations and use it to construct robust second-order scheme. The second order scheme is based on MUSCL reconstruction process. The stiff source terms are discretize using explicit and implicit

discretization. The source discretization is then combined with flux discretization using Strang splitting. The numerical scheme is then tested on several numerical examples.

In [65, 67, 68], Shu and Zhang have developed a positivity-preserving discontinuous Galerkin scheme for Euler equations of compressible flows. This is achieved by designing a limiter which ensures positivity of density and pressure at the quadrature points. Also, the proposed limiter maintains the overall accuracy of the scheme in smooth regions. The limiter is based on the convexity of the solution space and results in solving a quadratic equation, to identify the suitable state, which guarantees the positivity. This work is a generalization of maximum-principle limiter proposed by them in [66]. We extend the work of Shu and Zhang to Ten-Moment equations and develop a DG scheme, which guarantees the positivity of the symmetric pressure tensor in addition to positivity of the density. The recipe to achieve this is similar to their work. We first rewrite the DG scheme as a linear combination of finite volume scheme and then design a positivity preserving limiter which ensures the positivity. This results in solving a cubic equation. To remove oscillations at the discontinuities, we develop a TVB limiter based on characteristic variables. The work is further extended by considering the source terms used in [7].

Another extended MHD plasma flow model is Two-Fluid plasma flow model [1, 38, 47, 59, 60]. This model is a generalization of MHD model as it does not assume *quasi-neutrality* and allows different velocity and pressure for the ion and electron flows. This leads to a hyperbolic balance laws, where the ion and electron flows are modeled using Euler equations of compressible flows. These are coupled with Maxwell's equations for electric and magnetic field via a stiff source term, which models Lorentz force on the flow. In [59], the authors derive a Roe-type Riemann solver. Time updates are performed by treating the stiff source term implicitly and the flux terms explicitly. The resulting nonlinear equations are solved using Newton iterations. This method might be diffusive and may require many iterations for each time step. In [27], the author proposes a wave propagation methods for the spatial discretization. For time updates, a second-order operator splitting approach is used. A similar approach is taken in [46, 47], where spatial discretization is based on discontinuous Galerkin (DG) methods and time update is based on SSP-RK methods. Both of these approaches are easy to implement but

can be computationally expensive, especially for the realistic charge to mass ratios. More recently, in [38] authors have presented entropy stable schemes for the two-fluid equations. The source terms are treated implicitly, and resulting algebraic equations are solved exactly. However, schemes still do not guarantee the positivity of density and internal energy. More recently in [1] authors have designed a positivity preserving schemes for this model.

Our aim in this work is to design positivity preserving DG methods for Two-Fluid plasma flow equations. We use nodal basis on the Gauss-Lobatto-Legendre (GLL) nodes. The integrals are approximated by Gauss quadrature rule and Gauss-Lobatto quadrature of same GLL nodes, which are used to construct the nodal basis. The scheme for the Two-Fluid model is then obtained by combining the positivity preserving DG scheme for Euler equations for ion and electron, and a DG scheme for Maxwell equations. The source was treated using an implicit scheme to deal with the stiff source term. This ensures the positivity of the whole scheme.

The thesis is organized as follows:

In **Chapter 1**, we present the preliminaries for Hyperbolic PDEs, which includes the notion of hyperbolicity, weak solutions, and entropy condition. We also present some examples of popular hyperbolic systems and discuss the hyperbolicity and solution domain of these models. In **Chapter 2**, we briefly discuss finite volume methods for Hyperbolic PDEs. We present several approximated Riemann solver based numerical fluxes, followed by second-order MUSCL based second order schemes. In the end, higher order methods are discussed. We also present time discretization based on Runge-Kutta methods.

The first original contribution of the thesis is presented in **Chapter 3**, where we design positivity preserving FV methods for Ten-Moment Gaussian closure equations based model with source terms. We design two second-order positivity preserving reconstruction processes, which are based on the limiting of primitive variables. This is combined with positivity preserving implicit and explicit source discretizations, which results in a positivity preserving scheme for the complete system. In **Chapter 4**, we propose a well-balanced scheme for Ten-Moment equations with source terms. The well-balanced property is achieved by reconstruction of the scaled primitive variables and appropriate

source discretization. In **Chapter 5**, we extend the work of Chapter 3 to a plasma flow model based on Ten-Moment equations.

In the second part of the thesis, we consider DG schemes. In **Chapter 6**, we propose positivity preserving limiter for Ten-Moment equations, which ensures positivity of the DG scheme for Ten-Moment equations with source terms. In **Chapter 7**, we propose a DG positivity preserving scheme for Two-Fluid plasma flow model.