ON IDENTITIES WITH DERIVATIONS IN PRIME AND SEMIPRIME RINGS

CHIRAG GARG

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI

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ON IDENTITIES WITH DERIVATIONS IN PRIME AND SEMIPRIME RINGS

by

CHIRAG GARG

Department of Mathematics

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"On Identities with Derivations in Prime and Semiprime Rings"** submitted by **Mr. Chirag Garg** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

Dr. R. K. Sharma
Professor
Department of Mathematics
Indian Institute of Technology Delhi
New Delhi 110 016

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New Delhi *Chirag Garg*

Abstract

By a derivation, we mean an additive mapping d on a ring R such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Derivations on rings play a significant role in ring theory. In 1957, Posner proved a milestone result related to derivations on rings, and investigated the structure of rings and nature of mappings. A number of authors have extended Posner's result in various directions such as generalized derivation, Jordan derivation, (α, β)-derivation, generalized (α, β)-derivation, multiplicative derivation, multiplicative generalized derivation etc., on appropriate subsets of a ring such as ideal, left ideal, right ideal, Lie ideal, Jordan ideal, etc. In particular, motivated by the fact that a non central Lie ideal contains all commutators of a nonzero ideal of a ring with characterstic not 2, Wong generalized the Posner's result on multilinear polynomials. Afterwards, several authors studied different identities involving various mappings on multilinear polynomials. In this thesis, we study identities involving generalized derivations with multilinear polynomials on prime rings, and obtain the structure of mappings and rings.

There is a vast literature which studies identities on different derivations involving annihilating conditions. The seminal result in this direction was given by Posner who proved that if $a \in R$ and d is a derivation on prime ring R such that $ad(x) = 0$ for all $x \in R$, then either $a = 0$ or $d = 0$. Many authors extended this result in various directions. In this thesis, we study the identities involving generalized derivation on prime ring R with annihilating condition and obtain the commutativity of R. We also investigate annihilating condition of an identity involving automorphism on prime ring.

Jacobson in his book "Structure of Rings" has introduced the notion of $(s_1, s_2)-$derivation which was later more commonly known as (α, β)-derivation. In this direction, in this thesis we study the connection between the nature of generalized (α, β)-derivation on appropriate subsets of prime ring R and commutativity of R. If we remove the additivity property of generalized (α, β)-derivation G, then it is called multiplicative generalized (α, β)-derivation. Moreover, if we drop all restrictions on its associated map d as well, that is, if d is any map, then G is called multiplicative (generalized)-(α, β)-derivation. In this thesis, we prove that every multiplicative generalized (α, β)-derivation on R becomes additive under the existence of non-trivial idempotent element in R which satisfies some conditions. In this sequence, we also study the identities involving multiplicative (generalized)-(α, β)-derivation on left ideal of a prime ring and derive some important results.

सार

डेरिवेशन से हमारा अभिप्राय एक वलय R पर एक योगात्मक फलन d से है जो सभी x,yϵR के लिए d(xy) = d(x)y + xd(y) को संतुष्ट करता है। वलय सिद्धांतो में डेरिवेशन्स महत्वपूर्ण भूमिका निभाते हैं। सन् 1957 में, पोसनर ने वलयों पर डेरिवेशन्स से संबंधित एक मील का पत्थर परिणाम सिद्ध किया, और वलयों की संरचना एवं फलनों की प्रकृति की खोज की। कई लेखकों ने पोसनर के सिद्धांत का विभिन्न दिशाओं में विस्तार किया, जैसे वलय के उपयुक्त उपसमुच्चय जैसे आइडियल, बायां आइडियल, दायां आइडियल, ली आइडियल, जॉर्डन आइडियल आदि पर व्यापक डेरिवशन, जोर्डन डेरिवशन, (अल्फा, बीटा)-डेरिवशन, व्यापक (अल्फा, बीटा)-डेरिवशन, गुणात्मक डेरिवशन, गुणात्मक व्यापक डेरिवशन आदि। विशेष रूप से, तथ्य "एक गैर केन्द्रीय ली आइडियल एक वलय जिसकी characterstic 2 नहीं है के सभी गैर शून्य आइडियल के विनिमयों को सम्मलित करता है", से प्रेरित होकर वोंग ने पोसनर के सिद्धांत को बहुरेखिक बहुपदों पर सामान्यीकृत किया। इसके पश्चात, कई लेखकों ने बहुरेखिक बहुपदों पर विभिन्न फलनों को सम्मलित करने वाली विभिन्न सर्वसामिकाओं का अध्ययन किया। इस शोध ग्रंथ में, हम प्राइम वलयों पर बहुरेखिक बहुपदों के साथ व्यापक डेरिवेशन्स को सम्मलित करने वाली सर्वसामिकाओं का अध्ययन करते है, और फलनों और वलयों की सरंचना को प्राप्त करते हैं।

अन्निहिलटिंग परिस्थितियों को सम्मलित करने वाली विभिन्न डेरिवेशन्स पर सर्वसामिकाओं का अध्ययन करने का एक विशाल साहित्य उपलब्ध है। इस दिशा में एक फलवान सिद्धांत पोसनर ने दिया था जिसमें उन्होनें सिद्ध किया कि यदि एक प्राइम वलय R पर d एक डेरिवशन है और aϵR जो सभी xϵR के लिए ad(x)=0 को संतुष्ट करते हैं, तब या तो d=0 या a=0। अनेक लेखकों ने इस सिद्धांत का विभिन्न दिशाओं में विस्तार किया। इस शोध ग्रंथ में, हम अन्निहिलटिंग परिस्थिति के साथ प्राइम वलय R पर व्यापक डेरिवशन को सम्मलित करने वाली सर्वसामिकाओं का अध्ययन करते हैं, और वलय R की विनिमयता को प्राप्त करते हैं। हम प्राइम वलय पर ऑटोमोर्फिस्म को सम्मलित करने वाली सर्वसमिका की अन्निहिलटिंग परिस्थिति की खोज भी करते हैं।

जेकबसन ने अपनी पुस्तक "स्ट्रक्चर ऑफ रिंग्स" में (एस1, एस2)-डेरिवशन की धारणा का परिचय दिया है जोकि बाद में मुख्य रूप से (अल्फा, बीटा)-डेरिवशन के रुप में जाना गया । इस दिशा में, इस शोध ग्रंथ में हम प्राइम वलय R के उपयुक्त उपसमुच्चयों पर व्यापक (अल्फा, बीटा)-डेरिवशन की प्रकृति और R की विनिमयता के सम्बन्ध का अध्ययन करते हैं। यदि हम व्यापक (अल्फा, बीटा)-डेरिवशन G की योगात्मिक गुण को हटा देते हैं तब यह गुणात्मक व्यापक (अल्फा, बीटा)-डेरिवशन कहलाता है। इसके अतिरिक्त यदि हम सम्बंधित फलन d से सभी प्रतिबंधों से मुक्त कर देते है तब G एक गुणात्मक (व्यापक)-(अल्फा, बीटा)-डेरिवशन कहलाता है। इस शोध ग्रंथ में हम ने यह सिद्ध किया है कि वलय R में गैर तुच्छ इडेंपोटेंट तत्व के अस्तित्व के तहत जोकि कुछ शर्तों को संतुष्ट करता है, वलय R पर प्रत्येक गुणात्मक व्यापक

(अल्फा,बीटा)-डेरिवशन योगात्मक होता है। इस अनुक्रम में, हम एक प्राइम वलय के बायें आइडियल पर गुणात्मक (व्यापक)-(अल्फा, बीटा)-डेरिवशन को सम्मलित करने वाली सर्वसामिकाओं का अध्ययन भी करते हैं और कुछ महत्वपूर्ण सिद्धांत भी निकाले गए हैं।

Contents

List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{C}$	the set of complex numbers
$\mathbb{Z}_n$	the group of integers modulo n
$=$	equal to
$\neq$	not equal to
$s \in S$	s is a member of S
$s \notin S$	s is a member of S
$X \subseteq R$	X is a subset of R
$X \subset R$	X is a proper subset of R
$X \nsubseteq R$	X is not a subset of R
$\cup, \cap$	union, intersection
$A \times B$	cartesian product of A and B
$A \oplus B$	A direct sum B
$A \cong B$	A is isomorphic to B
$x \equiv y \pmod{m}$	x congruent to y modulo m
$< a >$	the cyclic group generated by a
S_n	permutation group on n symbol
$Z(R)$	center of a ring R

char(R)	characteristic of R
$l(S)$	left annihilator of S
$r(S)$	right annihilator of S
I	ideal of R
L	Lie ideal of R
I_l	left ideal of R
I_r	right ideal of R
I_d	dense right ideal of R
U	Utumi quotient ring
Q	Martindale ring of quotients
C	extended centroid of a prime ring R
GPI	generalized polynomial identity
soc(R)	socle of R