Parameter-Robust Numerical Methods for Singularly Perturbed Reaction-Diffusion Problems

By

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Department of Mathematics

Submitted in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

Indian Institute of Technology Delhi
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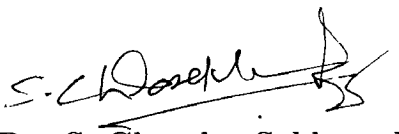
To

My Parents

Certificate

This is to certify that the thesis entitled **Parameter-Robust Numerical Methods for Singularly Perturbed Reaction-Diffusion Problems** submitted by **Mr. Mukesh Kumar** to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by him under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.



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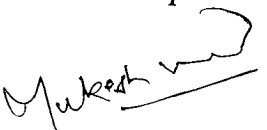

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Abstract

This thesis is devoted to design and analyze *Parameter-Robust Numerical Methods* in the spline collocation and the finite difference frameworks for *Singularly Perturbed Reaction-Diffusion Problems.*

To approximate the solution of stationary singularly perturbed reaction-diffusion problems three numerical methods are presented. Firstly, an Exponential B-Spline Collocation Method on uniform mesh is developed. The tension parameter in the construction of exponential B-spline basis function is so chosen that it captures the local behavior of the exact solution to the problem correctly. The error analysis is given and method is shown to have second order parameter-robust convergence. Secondly, an Optimal B-Spline Collocation Method is proposed in the domain decomposition framework. The essential idea in this method is to divide the domain of the differential equation into three non-overlapping subdomains and solve the regular problems obtained by transforming the differential equation with respective boundary conditions on these subdomains using the proposed B-spline collocation method. The error analysis of the method is given through matrix analysis approach and the optimal order convergence of the method is established. Finally, a High Order Parameter-Robust Numerical Method using a suitable combination of the fourth order compact difference scheme and the central difference scheme on the generalized Shishkin mesh $\widetilde{S}(L)$ is designed and the method is proved to have an almost fourth order parameter-robust convergence.

The idea of construction of High Order Parameter-Robust Numerical Method on the generalized Shishkin mesh is extended to singular perturbation problems of types; system of ($M \geqslant 2$) coupled reaction-diffusion problems, one-dimensional time dependent reaction-diffusion problems, two-dimensional time dependent reaction-diffusion problems, and system of ($M \geqslant 2$) coupled time dependent reaction-diffusion problems. A priori bounds on the solution are derived in the cases of system of ($M \geqslant 2$)-coupled stationary and time dependent singularly perturbed reaction-diffusion problems. The methods for all these problems are proved to have high order parameter-robust convergence.

Numerical experiments are conducted on some standard examples to validate the theoretical results.

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