ON CHANGES IN FROBENIUS NUMBER

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ON CHANGES IN FROBENIUS NUMBER

by

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To

My Parents

Rajinder and Kamlesh

In the memory of my sister

Certificate

This is to certify that the thesis entitled **On Changes In Frobenius Number** submitted by **Ms. Pooja Punyani** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi

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New Delhi *Pooja Punyani*

Abstract

Given a set $A = \{a_1, \ldots, a_k : \ a_i \in \mathbb{Z}_{\geq 0}\}$ of coprime positive integers, define $\Gamma(A) := \{a_1x_1 + \ldots a_kx_k : x_i \in \mathbb{Z}_{\geq 0}\}$. As $\Gamma^c(A) := \mathbb{N} \setminus \Gamma(A)$ is a finite set, we define

$$\mathtt{g}(A) := \max \Gamma^c(A), \ \mathtt{n}(A) := |\Gamma^c(A)|, \ \text{the sum } s(A) := \sum_{n \in \Gamma^c(A)} n.$$

Formulae for these functions are well known for the case $k = 2$. Let a 2-set be $\{a, b\}$, then we know

$$\mathtt{g}(a,b) = (a-1)(b-1)-1, \ \mathtt{n}(a,b) = \frac{1}{2}(a-1)(b-1), \ s(a,b) = \frac{1}{12}(a-1)(b-1)(2ab-a-b-1)$$

Using elementary tools, we characterize c for which $\mathtt{g}(a,b) - \mathtt{g}(a,b,c)$ belongs to the set $\{a, b, a+b, 2a, 2b, 2a+b, a+2b, 3a, 3b\}$ and in more generally, when the difference belongs to $\{ka, kb, ka+b, a+kb\}$ with some restrictions for k in case when diffrence is $ka+b$.

We also determine $\mathtt{g}(A), \mathtt{n}(A)$, $s(A)$ for certain families $A = \{a, b, c\}$. Along with this, for the same families A, we find the set $S^\star(A) = \{n \in \Gamma^c(A) : n + \Gamma(A)^\star \subset \Gamma(A)^\star\}$. Algorithmically, for general set $\{a, b, c\}$ we establish the difference $\mathbf{m}_{bi} - \mathbf{m}^\star_{bi}$ for each $i \in \{0, 1, \ldots, a\}$, where $\mathbf{m}_{bi}$ and $\mathbf{m}^\star_{bi}$ denote smallest integer congruent to bi mod a in $\Gamma(\{a, b\})$ and $\Gamma(\{a, b, c\})$, respectively. This enables us in giving a very neat formula for $\mathtt{n}(a,b,c), \mathtt{g}(a,b,c), s(a,b,c)$.

सार

हम एक दिए गए सह-अभाज्य पूर्णांकों के समुच्चय $A = \{a_1, \dots, a_k \colon a_i \ \in \mathbb{Z}_{\geq 0}\}$ को लेकर $\Gamma(A) \coloneqq \{a_1 x_1 + \cdots a_k x_k \colon x_i \in \mathbb{Z}_{\geq 0}\}$ को परिभाषित करते हैं। चूँकि $\Gamma^c(A) \coloneqq \mathbb{N} \setminus \Gamma(A)$ एक ससीम समुच्चय है, हम

$$g(A) \coloneqq \max \Gamma^c(A),$$

$$n(A) \coloneqq |\Gamma^c(A)|,$$

और

$$s(A) \coloneqq \sum_{n \in \Gamma^c(A)} n$$

और के जोड़ को परभाषित करते हैं।

$k = 2$ की स्थिति के सन्दर्भ में इन फलनों के लिए सूत्र विख्यात हैं। मानते हैं कि $\{a, b\}$ एक २-सेट है। फिर हमें पता है कि

$$\mathrm{g}(a,b) = (a-1)(b-1) - 1, \mathrm{n}(a,b) = \frac{1}{2}(a-1)(b-1), s(a,b) = \frac{1}{2}(a-1)(b-1)(2ab - a - b - 1)$$

बुनियादी साधनों का प्रयोग करके हम उस c का लक्षण वर्णन करते हैं जिसके लिए $g(a,b) - g(a,b,c)$ का अन्तर $\{a, b, a+b, 2a, 2b, 2a+b, a+2b, 3a, 3b\}$ के समूह का भाग है तथा और सामान्य रूप से अन्य समूहों का भाग प्रतीत होता है जब यह अन्तर $\{ka, kb, ka+b, a+kb\}$ का भाग है , जहाँ k पर कुछ प्रतिबन्ध है जब यह अन्तर $ka + b$ है।

हम $\mathrm{g}(A), \mathrm{n}(A), s(A)$ को कुछ निश्चित समाहार $A = \{a, b, c\}$ के लिए निर्धारित भी करते हैं। इसके साथ, उन्हीं समाहारों A के लिए, हम $S^\star(A) = \{n \in \Gamma^c(A) \colon n + \Gamma(A)^\star \subset \Gamma(A)^\star$ के समुच्चय को निर्धारित करते हैं। कलन विधि अनुसार, हम सामान्य समुच्चय $\{a, b, c\}$ के लिए $m_{bi} - m_{bi}^\star$ के अन्तर को हर $i \in \{0, 1, \dots, a\}$ के लिए स्थापित करते हैं, जहां m_{bi}और $m_{bi}^\star$ उन न्यूनतम पूर्णांकों को चिह्नित करते है जो क्रमशः $\Gamma(\{a, b\})$ और $\Gamma(\{a, b, c\})$ में $bi \ mod \ a$ के सर्वांगसम हैं। यह हमें $\mathrm{n}(a,b,c), \mathrm{g}(a,b,c), s(a,b,c)$ के लिए एक स्पष्ट सूत्र देने में सक्षम करता है।

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List of Symbols

Symbol	**Meaning**
A	set of coprime positive integers
$\mathbb{Z}$	set of integers
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}_{\geq 0}$	set of non negative integers
$\gcd(A)$	greatest common divisor of elements of A
$\Gamma(A)$	$\{a_1x_1 + \ldots a_kx_k : x_i \in \mathbb{Z}_{\geq 0}\}$
$\Gamma^c(A)$	complement of $\Gamma(A)$ in $\mathbb{N}$
$\Gamma(A)^\star$	$\Gamma(A) \setminus \{0\}$
$\mathtt{g}(A)$	largest element of $\Gamma^c(A)$
$\mathtt{n}(A)$	cardinality of $\Gamma^c(A)$
$s(A)$	sum of elements of $\Gamma^c(A)$
$S^\star(A)$	$\{n \in \Gamma^c(A) : n + \Gamma(A)^\star \subset \Gamma(A)^\star\}$
$\mathbf{m}_i$	smallest integer congruent to i mod a in $\Gamma(\{a,b\})$
$\mathbf{m}_i^\star$	smallest integer congruent to i mod a in $\Gamma(\{a,b,c\})$