A

THESIS ON

SOME PLANE, TORSION AND DYNAMIC PROBLEMS OF ELASTICITY

BY

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K.L. Chowdhury 13.9.66.
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CERTIFICATE

This is to certify that the thesis entitled 'Some Plane, Torsion and Dynamic Problems of Elasticity' that is being submitted by Shri K.L.Chowdhury for the award of the Degree of Doctor of Philosophy to Indian Institute of Technology, Delhi is a record of bonafide research work carried out by him under my supervision and guidance. Shri K.L. Chowdhury has worked for four years in the Department of Mathematics, Indian Institute of Technology, Delhi and the thesis has reached the standard fulfilling the requirements of the regulations relating to the Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any Degree or Diploma.

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SYNOPSIS

This thesis, divided in three parts, comprises the solution of some problems in (i) Rotation of blades (ii) Torsion of bars (iii) Elastodynamics. The closed form solutions for the generalized plane stress and torsion problems are obtained using complex potential theory and conformal mapping. Problems of elastodynamics have been solved by the application of Laplace transforms and Cagniard's technique and frequent use has been made of integral representations of Bessel functions and their properties to find the inverse transforms.

Part A

Rotation of Blades

This part is concerned with the generalized plane stress problems of finding stresses in certain thin curvilinear isotropic elastic blades rotating steadily about axes within or perpendicular to their planes. The stress and displacement combinations are expressed in terms of two functions of a complex variable, a body force potential and a mapping function. The mapping function maps the region within the blade in z-plane onto the interior of unit circle, in ζ-plane. The tentative method of solution consists in assuming suitable forms for complex potentials and to evaluate the constants involved from the boundary condition satisfied by them. The forms of the body force

potential and mapping function vary with the position of axis of rotation and the shape of the blade respectively. The following problems have been considered:-

1. Curvilinear elastic blades rotating steadily about a normal axis passing through the centre:

The region within the blade in z-plane is mapped on the unit circle in ζ-plane by the mapping function

$$z = \frac{c\zeta}{1+m\zeta^{p+1}} \quad c > 0, \ |m| < \frac{1}{p}, \ \zeta = \rho e^{i\theta}$$

where c, m and p are real constants, p being a positive integer. The closed form expressions for complex potentials (involving integrals, which are evaluated) and hoop-stress are obtained. The hoop-stress is found symmetrical about the lines $\theta = \frac{n\pi}{p+1}$ and it is maximum at $\theta = \frac{2n\pi}{p+1}$ and minimum at $\theta = \frac{(2n+1)\pi}{p+1}$ where $n = 0, 1, 2, 3 \ldots\ldots p$. The following particular cases are discussed in detail:-

(a) $p = 1$, the blade is of Booth's Lemniscate shape

(b) $p = 2$, the shape is inverse of a curvilinear equilateral triangle with respect to the circle $|z| = c$

(c) $p = 3$, the shape of the blade is inverse with respect to the circle $|z| = c$ of a curvilinear square.

For (a), the variation of hoop-stress from $\theta = 0$ to $\theta = \frac{\pi}{2}$ for $m = 0.5, 0.4, 0.3$ and of maximum hoop-stress for different values of $0 < m < 1$ is tabulated and illustrated by graphs. It is concluded that hoop-stress decreases continuously from maximum value to minimum value for all values of the parameter $0 < m < 1$ and the maximum hoop-stress increases continuously with the parameter. For (b) and (c) the variation of maximum hoop-stress for different values of $0 < m < 0.4$ is tabulated and illustrated by a graph. Another figure illustrates the variation of hoop-stress for $p = 2$; $m = 0.2, 0.3$ and $p = 3$; $m = 0.1, 0.2$. In all the cases the maximum and minimum hoop-stresses are found to occur at points closest and farthest from the axis of rotation respectively.

2. (a) Booth's Lemniscate shaped blade rotating with uniform angular velocity about the axes of symmetry in its plane

   (b) Blade bounded by a loop of Bernoulli's Lemniscate rotating about the axis of symmetry.

   The complex potentials and hoop-stresses are found in closed forms, for both (a) and (b). For (a), variation of hoop-stress for different values of m at stationary points $\theta = 0$ and $\theta = \frac{\pi}{2}$ is tabulated and graphed. A point of maxima in $0 < \theta < \frac{\pi}{2}$ is found to vary with m. For (b) the numerical values of hoop-stress are tabulated and illustrated by a graph.

3. (a) Limacon shaped blade:-

   The exact closed form expressions are obtained for complex potentials and hoop-stresses for the following four positions of axis of rotation:-

   (i) the line of symmetry of Limacon.

   (ii) Any line perpendicular to the line of symmetry and lying in the plane of the blade.

(iii) Any line passing through a point on the line of symmetry and perpendicular to the plane of the blade.

(iv) Normal axis passing through the centroid.

In general for (ii), (iii) the body forces are found to have a resultant due to non-balance of inertia forces. The hoop-stress and its stationary points have been calculated in all cases. Numerical data is illustrated by graphs to show variation of hoop-stress and maximum hoop-stress in cases (i) and (iv).

(b) Results are derived from (ii) and (iii) for circular blade rotating steadily about:-

(i) a chord (ii) a normal axis passing through any point within or on the boundary respectively. The difference in hoop-stresses in normal and chord cases is found constant and is equal to the hoop-stress of a blade rotating about a diameter. The numerical values of hoop-stress are contained in tables and illustrated by graphs.

Part B

Torsion of Bars

This part deals with Saint Venant's problem of torsion of isotropic elastic cylinders. We give a short account of the general theory and the basic equations. The region within the cross-section in z-plane is mapped conformally onto a semi-circle in ζ-plane. Considered in detail are the following problems:

1. Elastic cylinder with regular curvilinear semi-polygonal cross-section:

The expressions in closed form are obtained for the complex torsion function, torsional rigidity, shearing and peripheral stress for an elastic bar of regular curvilinear semi-polygonal cross-section whose region can be mapped conformally onto the interior of semi-circle in ζ-plane by the mapping function

$$z = c(\zeta + m\zeta^{n}) \ , \quad 0 \leqslant |\, mn \,| < 1$$

where c and m are real constants and n is a positive integer. The results for the following different shapes of cross-sections are derived:-

(a)	Semi circle	$(m = 0)$
(b)	(i) Semi-Elliptic Limacon	$(n = 2, \ 0 \leqslant m < \frac{1}{2})$
	(ii) Semi Cardiod	$(n = 2, \ m = \frac{1}{2})$
(c)	Semi-dumb-bell	$(n = 3, \ \lvert m \rvert < \frac{1}{3})$
(d)	Right-angled triangle with 30^0 and 60^0 angles	$(n = 4, \ \lvert m \rvert < \frac{1}{4})$
(e)	(i) Isosceles right-angled triangle	$(n = 5, \ 0 \leqslant m < \frac{1}{5})$
	(ii) Rectangle with one side half the other	$(n = 5, \ -\frac{1}{5} < m \leqslant 0)$

For (a) and (b) results found agree with those already known. The variation of torsional rigidity for

(i) $n = 2, \ 0 < m < \frac{1}{2}$

(ii) $n = N, \ -\frac{1}{N} \leqslant m < \frac{1}{N} \qquad (N = 3, 4, 5)$

are shown in tables and numerical data is illustrated by a graph. For (i) torsional rigidity is found increasing with m.

2. Elastic cylinder whose cross-section is bounded by semi-inverse of an ellipse with respect to any point on the major or minor axis:

The exact closed form expression for the torsion function is obtained for the isotropic elastic cylinder whose region within its boundary in z-plane is mapped conformally onto the interior of semi-circle by the mapping function

$$z = \frac{c\zeta}{1+m\zeta+p\zeta^2}, \quad |\, p \,| < 1, \quad |\, m \,| < 2$$

m, c and p being real constants. Taking $m = 0$, the shape of the cross-section becomes a semi-Booth's Lemniscate. The expressions for torsional rigidity and shearing and peripheral stresses are derived.

Part C

Elasto-dynamics

The following problems are discussed in this part:

1. Dynamic stresses in a rotating cylinder

Dynamic stresses are set up when a solid or hollow isotropic elastic cylinder rotates with variable angular velocity. The equations of motion have been written down by including the reversed mass acceleration terms. The solution of equations of motion, for any arbitrary angular velocity is obtained making use of Laplace transforms, Bessel functions and their properties and Cauchy's residue theory. The equations of motion have also been solved by the method of separation of variables when the angular velocity has a Fourier series expansion. In the first case, displacements and stresses are expressed in the integral form, and in the second case these are expressed as infinite series.

The following specific examples are discussed.

(i) Sudden motion of a solid cylinder

(ii) Solid cylinder moving with periodic angular velocity.

2. Dynamic Thermoelastic problems of

(i) a cylinder

(ii) infinite medium with a cylindrical hole.

The stress in (i) or ?(ii) results from the temperature applied on the boundary surface. Initially the medium is assumed to be undisturbed and at zero temperature. Inertia effects are included in equations of motion. Solution is developed through the method of variation of parameters and Laplace transforms. Bessel's function and their properties and Cauchy's residue theorem have been used for evaluation of inversion integrals. From the general solution, the solutions for two specific forms of temperature are derived. Expression for displacement at the surface, for small values of time, in case of sudden heating of solid cylinder, has been obtained using asymptotic expansions of Bessel functions.

3. Time-dependent thermal stresses in an Elastic layer :

Laplace's transform technique is employed to obtain solution of dynamic thermoelastic equations for an infinite isotropic elastic layer of thickness h resting on a rigid foundation. The expressions for displacements and stresses are worked out. The specific example of constant temperature suddenly applied to the upper surface of elastic layer is examined. An inertia parameter is introduced and solution for the quasi-static case is obtained by letting it approach zero.

4. Disturbance produced in an elastic half space due to normal pressure on a circular portion of its boundary.

The normal pressure on the boundary surface $z = 0$, in the general case, is specified by

$$\begin{aligned} \widehat{zz} &= R(r)\,P(t) \qquad && (0 \leqslant r \leqslant a) \\ &= 0 && (r > a) \end{aligned}$$

where R(r) and P(t) are arbitrary functions of r and of time t respectively. Laplace and Hankel transforms are applied to the equations of motion. For two specific forms of $\widehat{zz}(r,t)$, displacement components are obtained in the forms of definite integrals. Limiting processes determine the surface displacements which have been split up in terms representing P-wave and Rayleigh wave contributions. Solution is obtained using Cagniard's technique, residue theory and convolution theorem.

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