# CONTRIBUTION TO THE STUDY OF POLYNOMIALS AND ENTIRE FUNCTIONS

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## CERTIFICATE

This is to certify that the thesis entitled "Contribution to the study of polynomials and entire functions" which is being submitted by Mr. V.K. Jain for the award of Doctor of Philosophy in Mathematics to the Indian Institute of Technology, New Delhi, is a record of bonafide research work. He has worked for the past four and half years under my guidance and supervision.

The thesis has reached the standard fulfilling the requirements of regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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## SYNOPSIS

This thesis consists of three chapters. The first chapter concerns with the location of the zeros of a polynomial. In the second chapter we study the extremal properties of the polynomials and in the third chapter analogous problems for entire functions have been considered.

Let $p(z) = \sum_{\nu=0}^{n} a_{\nu} z^{\nu}$ be a polynomial of degree n. If $a_{\nu}$'s are real, non-negative and monotonic non-decreasing then according to well known Eneström-Kakeya theorem, p(z) has all its zeros in the closed unit disc. This is a very elegant result but it is equally limited in scope. The hypothesis is very restrictive and does not seem useful for applications. Various generalisations of this result have been obtained, among others by G.T. Cargo and O.Shisha [13], A. Joyal, G. Labelle and Q.I. Rahman [33], P.V. Krishnaiah [37], Rubinstein [58] and Govil and Rahman [29]. In the first chapter we obtain an improvement of the result of Govil and Rahman [29] by obtaining the zero free region bigger than the zero free region obtained by Govil and Rahman [29]. Several other related results have also been obtained.

If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, then according to Bernstein's theorem

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)| ,$$

$$\int_0^{2\pi} |p'(e^{i\theta})|^2 \, d\theta \leq n^2 \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta \quad \text{and}$$

$$\int_0^{2\pi} |p(Re^{i\theta})|^2 \, d\theta \leq R^{2n} \int_0^{2\pi} |p(e^{i\theta})|^2 \, d\theta , \quad R > 1.$$

P.D. Lax [39] has considered the class of polynomials $p(z)$ having no zero in the interior of the unit disc. Rahman [56] considered the class of polynomials having no zero in $|z| < k$, $k \leq 1$. For polynomials having no zero in $|z| < k$, $k \geq 1$, Malik [41] has proved

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|$$

(Also see Govil and Rahman [27]). Recently Giroux and Rahman [26] have considered the class of polynomials $p(z)$ satisfying the condition $p'(1) = 0$. In the second chapter we consider the class of polynomials $p(z)$ satisfying $p(z) \equiv z^n \, p(\frac{1}{z})$. We have not been able to solve the problem completely. The class of polynomials $p(z) \equiv z^n \, p(\frac{1}{z})$

is interesting in view of the fact that if $p(z)$ is any polynomial of degree n then $P(z) = z^n p(z + \frac{1}{z})$ is a polynomial of degree 2n satisfying the condition $P(z) \equiv z^{2n} P(\frac{1}{z})$. Besides the class of polynomials satisfying $p(z) \equiv z^n p(\frac{1}{z})$, some other classes have also been considered.

If $f(z)$ is an entire function of exponential type $\tau$ such that $|f(x)| \leqslant M$ on the real axis, then $|f'(x)| \leqslant M\tau$, $|f(x+iy)| \leqslant Me^{\tau|y|}$, $-\infty < x < \infty$, $-\infty < y < \infty$. Boas [9] considered the class of entire functions $f(z)$ of exponential type $\tau$ having no zero in $y > 0$. Govil and Rahman [27] considered the class of entire functions $f(z)$ having no zero in $y > k$, $k \leqslant 0$. In third chapter we consider the class of entire functions having no zero in $y > k$, $k > 0$. We have not been able to obtain the sharp result. Also, Govil and Rahman [27] while considering the case $k \leqslant 0$ gave the bound for $|f(x+iy)|$ in the range, $2k < y \leqslant 0$, $-\infty < x < \infty$. We have been able to obtain a bound for $|f(x+iy)|$ for the entire range $-\infty < x < \infty$, $-\infty < y < 0$. Besides these some other related results have also been obtained.

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