

**STOCHASTIC FINITE ELEMENT MODEL
UPDATING USING MODAL DATA**

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**DEPARTMENT OF MECHANICAL ENGINEERING
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by

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DEPARTMENT OF MECHANICAL ENGINEERING

submitted

**in fulfilment of the requirements of the degree of doctor of philosophy
to the**



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Dedicated to My Parents

Late Mr. Chandra Mohan Sharma

&

Mrs. Uma Sharma

Certificate

This is to certify that the thesis entitled **Stochastic Finite Element Model Updating using Modal Data** being submitted by Nikhil Sharma to the Indian Institute of Technology Delhi for the award of the degree of Doctor of Philosophy is a record of bonafide research carried out by him under my supervision and guidance the thesis work, in my opinion, has reached the requisite standard fulfilling the requirement for the degree of Doctor of Philosophy. The results contained in this thesis have not been submitted in part or in full to any other university or institute for the award of any degree or diploma.

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Nikhil Sharma

Abstract

This thesis deals with studies in stochastic FE model updating. The scope of the studies includes developing improved stochastic model updating methods which can also identify damping matrix variability in addition to mass and stiffness matrix variability. Initially, a detailed comparative study of the stochastic model updating methods, namely minimum variance methods, perturbation method, and Monte Carlo Simulation method, is undertaken with the objective of studying their relative advantages and limitations. Samples in Monte Carlo simulation are generated with the Normal and Latin Hypercube Sampling method. Case study of a cantilever beam structure using computer-simulated experimental data is considered for this purpose. On the basis of this study, a robust methodology to deal with measurement noise is proposed by carrying out multiple modal tests on each sample, thereby allowing to obtain an unbiased estimate of the mean dynamic characteristics for each sample.

A method that can allow taking into account any known prior information about the maximum allowable variation in mean and covariance of the uncertain parameters is needed. This would allow obtaining a physically meaningful updated stochastic FE model. To meet this objective, a constrained optimization-based stochastic model updating method is proposed. In the constrained optimization approach, any known prior information about the maximum allowable variation in mean and covariance of the uncertain parameters is incorporated via lower and upper bounds on the mean and covariances in the form of inequality constraints. The performance of the proposed method is also compared with an existing perturbation method of stochastic model updating through two numerical studies, one on cantilever beams and another on crossed-beam structures. The result obtained by the proposed method shows that the method

is effective in identifying uncertainties in the local region as well as those existing at the component or global level.

It is observed from the literature that though there are studies on the effects of measurement noise on the estimates of the variability of the updating parameters, but there are no approaches available to estimate the noise in the estimates of variabilities. If the noise can be estimated, then a more accurate estimate of the covariance of the parameters can be found. Towards this objective, an approach for estimating measurement noise along with identifying the variability of mass and stiffness parameters is proposed. The performance of the proposed method is evaluated through numerical studies on crossed beam structures.

In most of the stochastic FE model updating methods, attention is given to identifying variability related to mass and stiffness matrices. But, in addition to mass and stiffness variability, the damping properties of a practical structure would also have variability. Modelling damping variability is essential for predicting the variability of dynamic response. Towards this objective, a new perturbation-based approach is proposed using statistics of complex modal data. The performance of the proposed method is evaluated through two numerical studies.

For the experimental validation of the proposed methods, two experimental studies are conducted. The first study is carried out on samples of nominally identical free-free beams. The second experimental study is carried out on samples of nominally identical crossed-beam structures. The results obtained from the experimental studies show that the proposed methods are able to identify and predict the variability of dynamic characteristics associated with samples of the structure.

सार

यह थीसिस स्टोकेस्टिक एफई मॉडल अपडेटिंग में अध्ययन से संबंधित है। अध्ययन के दायरे में बेहतर स्टोकेस्टिक मॉडल अपडेटिंग विधियों को विकसित करना शामिल है जो द्रव्यमान और स्तिफनेस मैट्रिक्स परिवर्तनशीलता के अलावा डैम्पिंग मैट्रिक्स परिवर्तनशीलता की भी पहचान कर सकते हैं। प्रारंभ में, स्टोकेस्टिक मॉडल अपडेटिंग विधियों, जैसे मिनिमम वैरिअन्स विधि, पटरबेशन विधि और मोंटे कार्लो सिमुलेशन विधि का एक विस्तृत तुलनात्मक अध्ययन उनके सापेक्ष फायदे और सीमाओं का अध्ययन करने के उद्देश्य से किया गया है। मोंटे कार्लो सिमुलेशन में नमूने सामान्य और लैटिन हाइपरक्यूब सैंपलिंग विधि से तैयार किए जाते हैं। इस उद्देश्य के लिए कंप्यूटर-सिम्युलेटेड प्रायोगिक डेटा का उपयोग करके कैंटिलीवर बीम संरचना के केस अध्ययन पर विचार किया गया है। इस अध्ययन के आधार पर प्रत्येक नमूने पर कई मोडल परीक्षण करके माप शोर से निपटने के लिए एक मजबूत पद्धति प्रस्तावित की है जिससे प्रत्येक नमूने के लिए औसत गतिशील विशेषताओं का निष्पक्ष अनुमान प्राप्त करने की अनुमति मिलती है।

एक ऐसी विधि की आवश्यकता है जो माध्य में अधिकतम स्वीकार्य भिन्नता और अनिश्चित मापदंडों के सहप्रसरण के बारे में किसी भी ज्ञात पूर्व जानकारी को ध्यान में रख सके। यह एक भौतिक रूप से सार्थक अपडेटेड स्टोकेस्टिक एफई मॉडल प्राप्त करने की अनुमति देगा। इस उद्देश्य को पूरा करने के लिए एक कन्स्ट्रैनेड ऑप्टिमाइजेशन आधारित स्टोकेस्टिक मॉडल अपडेटिंग विधि प्रस्तावित की है। कन्स्ट्रैनेड ऑप्टिमाइजेशन दृष्टिकोण में अनिश्चित मापदंडों के माध्य और सहप्रसरण में अधिकतम स्वीकार्य भिन्नता के बारे में किसी भी ज्ञात पूर्व जानकारी को इनक्वूलिटी कंस्ट्रेंट्स के रूप में माध्य और सहप्रसरण पर निचली और ऊपरी सीमाओं के माध्यम से शामिल किया जाता है। प्रस्तावित विधि के प्रदर्शन की तुलना दो संख्यात्मक अध्ययनों के माध्यम से मौजूदा पटरबेशन विधि से भी की गयी है। एक कैंटिलीवर बीम संरचनाओं पर और दूसरा क्रॉस-बीम संरचनाओं पर। प्रस्तावित विधि द्वारा प्राप्त

परिणाम से पता चलता है कि यह विधि स्थानीय क्षेत्र के साथ-साथ घटक या वैश्विक स्तर पर मौजूद अनिश्चितताओं की पहचान करने में प्रभावी है।

साहित्य से यह देखा गया है कि यद्यपि अपडेटेड मापदंडों की परिवर्तनशीलता के अनुमानों पर माप शोर के प्रभावों का अध्ययन है, लेकिन परिवर्तनशीलता के अनुमानों में शोर का अनुमान लगाने के लिए कोई विधि उपलब्ध नहीं है। यदि शोर का अनुमान लगाया जा सकता है, तो मापदंडों के सहप्रसरण का अधिक सटीक अनुमान पाया जा सकता है। इस उद्देश्य की प्राप्ति के लिए, द्रव्यमान और स्टीफनेस मापदंडों की परिवर्तनशीलता की पहचान के साथ-साथ माप शोर का अनुमान लगाने के लिए एक विधि प्रस्तावित की है। प्रस्तावित विधि के प्रदर्शन का मूल्यांकन क्रॉस बीम संरचनाओं पर संख्यात्मक अध्ययन के माध्यम से किया गया है।

अधिकांश स्टोकेस्टिक एफई मॉडल अपडेटिंग विधियों में, द्रव्यमान और स्टीफनेस मैट्रिक्स से संबंधित परिवर्तनशीलता की पहचान करने पर ध्यान दिया गया है। लेकिन, द्रव्यमान और स्टीफनेस परिवर्तनशीलता के अलावा, व्यावहारिक संरचना के डैम्पिंग गुणों में भी परिवर्तनशीलता होती है। डायनामिक रिस्पांस की परिवर्तनशीलता की भविष्यवाणी के लिए डैम्पिंग परिवर्तनशीलता की मॉडलिंग भी आवश्यक है। इस उद्देश्य की प्राप्ति के लिए, काम्प्लेक्स मोडल डेटा के आँकड़ों का उपयोग करके एक नया पटरबेशन आधारित दृष्टिकोण प्रस्तावित किया गया है। प्रस्तावित पद्धति के प्रदर्शन का मूल्यांकन दो संख्यात्मक अध्ययनों के माध्यम से किया गया है।

प्रस्तावित विधियों के प्रयोगात्मक सत्यापन के लिए, दो प्रयोगात्मक अध्ययन आयोजित किए गए हैं। पहला अध्ययन नोमिनल समान मुक्त-मुक्त बीम के नमूनों पर किया गया है। दूसरा प्रायोगिक अध्ययन नोमिनल समान क्रॉस-बीम संरचनाओं के नमूनों पर किया गया है। प्रायोगिक अध्ययनों से प्राप्त परिणाम बताते हैं कि प्रस्तावित विधियाँ संरचना के नमूनों से जुड़ी डायनामिक विशेषताओं की परिवर्तनशीलता की पहचान और भविष्यवाणी करने में सक्षम हैं।

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List of symbols and abbreviations

Symbols

z	Modal data
θ	Updating parameter
$z(\theta)$	Analytical Modal data
$[\]$	Matrix
$\{ \}$	Vector
$-$	Denote Mean values
$\wedge$	Denote Normalized terms
Δ	Change in parameters
$\odot$	Hadamard product
$[M]$	Mass Matrix
$[K]$	Stiffness Matrix
$[D]$	Damping Matrix
$[S]$	Modal Sensitivity Matrix
$[T]$	Transformation Matrix
D_c	Correlation Matrix
E	Operator used for the calculation of Expected value
cov	Operator used for the estimation of covariances
Y	Random part in measured modal data
$\{\varepsilon\}$	Residue vector
$\{\lambda\}$	Eigenvalues vector
$[\phi]$	Eigenvector
$[\Sigma_\theta]$	Covariance matrix of the updating parameters
σ_θ	Standard deviation of the updating parameters
$\{\Psi\}$	Vector of random measurement noise
$[V], [C]$	Covariance (or Variance) Matrix
Q	Number of test structures
$\{p\}, \{u\}$	Vector of fractional correction factors of updating parameters

$J_1(p)$	Objective function for updating the mean values of updating parameters
$J_2(\text{COV})$	Objective function for updating COV's of updating parameters
$[H]$	Covariance matrix of the eigenvalues due to the structural variability
$[O]$	Covariance matrix of the eigenvalues due to the measurement noise
$[\Sigma_\theta]$	Covariance matrix of the updating parameters
ω	Frequency in rad./sec.
f	Frequency in Hz
β	Coefficient of Damping Matrix
η	Loss Factor

Subscripts

A	Analytical or FE model
X	Experimental model or data
m	Mean values of stochastic elements
p	Random values of stochastic elements

Abbreviations

FE	Finite Element
COV	Coefficient of variation of the updating parameters
COV_{noise}	Coefficient of variation of noise
FRF	Frequency Response Function
MCS	Monte Carlo Simulation
Re	Real Part
Im	Imaginary Part