

**ENTROPY STABLE NUMERICAL SCHEMES FOR
HYPERBOLIC BALANCE LAWS**

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ENTROPY STABLE NUMERICAL SCHEMES FOR HYPERBOLIC BALANCE LAWS

by

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Submitted

in fulfillment of the requirements of the degree of Doctor of Philosophy

to the



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Dedicated to
My Grandparents

Certificate

This is to certify that the thesis titled **Entropy Stable Numerical Schemes For Hyperbolic Balance Laws** submitted by **Ms. Chhanda Sen** to the Indian Institute of Technology, Delhi, for the award of the degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

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Abstract

Systems of Hyperbolic conservation laws appear in the modeling of several engineering, physical and biological phenomenon. Due to this wide applicability, there is great interest in theory and simulations of these systems. However, theory for the related initial value problem is not developed especially in higher dimensions. Due to this, simulations are the preferred option. Still, it is preferable to have the numerical result which has the same stability property of the continuous problem.

Due to often present nonlinearity in the flux, in general, the smooth solution does not exist for all time. Due to this, we need to consider the weak formulation of the system which leads to non-unique solutions even in the scalar case. To choose the physically relevant solution, a criterion in the form of entropy condition is imposed. This is one of the very few stability estimates available for the systems in higher dimensions. Even then the solution may not be unique for the systems of conservation laws. Still, it is desirable for numerical solutions to satisfy a discrete version of the entropy estimate. This is the focus of the thesis. We aim to design entropy stable schemes which are higher order accurate for Ten-Moment fluid flow model, Extended MHD plasma flow model, and Two-Fluid Ten-Moment plasma flow model.

Ten-Moment Gaussian closure fluid flow equations model fluid flow without the assumption of local thermodynamic equilibrium. This leads to a tensorial description of pressure which is distinctively different from the Euler equations for compressible flows. We design second and third-order accurate entropy stable schemes for the system. We construct an approximate entropy conservative flux in addition to an entropy conservative flux. Both of them are kinetic energy preserving. Higher order sign preserving MinMod

and ENO interpolations are used to design higher order entropy diffusion operator.

Next, we consider a recently proposed extended magnetohydrodynamics (XMHD) plasma flow model where electron pressure, electron inertia, and Hall term effects are also considered. The resulting model is a more suitable description of the high energy density plasmas. In this work, we design second order entropy stable finite difference schemes for the model by exploiting the structure of the flux. These schemes are based on the entropy stable schemes for the Euler equations. We also propose an implicit discretization of the source, which overcomes time step restriction due to the stiff source. The resulting algebraic system is then solved exactly, which results in a computationally efficient implementation.

In the last contribution of the thesis, we consider a Two-Fluid plasma flow balance law, where Ten-Moment equations in three dimension model fluid components. We develop second-order entropy stable numerical schemes for the model. To achieve this, we first construct an entropy conservative flux which is modified suitably by addition of entropy diffusion term. The diffusion term is constructed using entropy scaled eigenvectors. Furthermore, source terms are treated using both explicit and implicit discretizations.

सार

हाइपरबोलिक संरक्षण कानूनों की प्रणालियाँ कई इंजीनियरिंग के मॉडलिंग में दिखाई देती हैं, भौतिक और जैविक घटना। इस व्यापक प्रयोज्यता के कारण, बड़ी रुचि है इन प्रणालियों के सिद्धांत और सिमुलेशन में। हालांकि, संबंधित प्रारंभिक मूल्य के लिए सिद्धांत समस्या विशेष रूप से उच्च आयामों में विकसित नहीं होती है। इसके कारण सिमुलेशन हैं पसंदीदा विकल्प। फिर भी, संख्यात्मक परिणाम होना बेहतर है, जो कि है निरंतर समस्या की एक ही स्थिरता संपत्ति।

अक्सर प्रवाह में गैर-मौजूदता के कारण, सामान्य रूप से, चिकनी समाधान नहीं होता है हर समय मौजूद है। इसके कारण, हमें सिस्टम के कमजोर निर्माण पर विचार करने की आवश्यकता है जो स्केलर मामले में भी गैर-अनूठे समाधान की ओर जाता है। शारीरिक रूप से चुनने के लिए प्रासंगिक समाधान, एन्ट्रापी स्थिति के रूप में एक मानदंड लगाया जाता है। यह वही है उच्च आयामों में सिस्टम के लिए बहुत कम स्थिरता के अनुमान उपलब्ध हैं। यहाँ तक की तब समाधान संरक्षण कानूनों की प्रणालियों के लिए अद्वितीय नहीं हो सकता है। फिर भी, यह है एन्ट्रापी अनुमान के एक असतत संस्करण को संतुष्ट करने के लिए संख्यात्मक समाधानों के लिए वांछनीय है। यह थीसिस का फोकस है। हम एंट्रॉपी स्थिर योजनाओं को डिजाइन करना चाहते हैं जो अधिक हैं टेन-मोमेंट द्रव प्रवाह मॉडल, विस्तारित MHD प्लाज्मा प्रवाह मॉडल, और के लिए सटीक क्रम दो-द्रव दस-पल प्लाज्मा प्रवाह मॉडल।

टेन-मोमेंट गौसियन बंद तरल पदार्थ प्रवाह समीकरण मॉडल द्रव प्रवाह धारणा के बिना स्थानीय थर्मोडायनामिक संतुलन। यह एक तन्वता वर्णन की ओर जाता है दबाव जो अलग-अलग प्रवाह के लिए यूलर समीकरणों से अलग है। हम सिस्टम के लिए दूसरे और तीसरे क्रम के सटीक एन्ट्रापी स्थिर योजनाओं को डिजाइन करते हैं। हम एक एन्ट्रापी रूढ़िवादी के अलावा एक अनुमानित एन्ट्रापी रूढ़िवादी प्रवाह का निर्माण प्रवाह। वे दोनों गतिज ऊर्जा संरक्षण हैं। उच्च क्रम साइन मिनमोड को संरक्षित करता है और ENO प्रक्षेप का उपयोग उच्च क्रम एन्ट्रापी डिफ्यूजन ऑपरेटर को डिजाइन करने के लिए किया जाता है।

अगला, हम हाल ही में प्रस्तावित विस्तारित मैग्नेटोहाइड्रोडायनामिक्स (एक्सएमएचडी) पर विचार करते हैं प्लाज्मा प्रवाह मॉडल जहां इलेक्ट्रॉन दबाव, इलेक्ट्रॉन जड़ता और हॉल शब्द प्रभाव होते हैं भी माना जाता है। परिणामी मॉडल उच्च ऊर्जा का अधिक उपयुक्त वर्णन है घनत्व plasmas। इस काम में, हम दूसरे क्रम एंट्रॉपी स्थिर परिमित अंतर को डिजाइन करते हैं प्रवाह की संरचना का शोषण करके मॉडल के लिए योजनाएं। ये योजनाएं आधारित हैं यूलर समीकरणों के लिए एन्ट्रापी स्थिर योजनाओं पर। हम एक निहित का प्रस्ताव भी करते हैं स्रोत का विवेकाधिकार, जो कठोर स्रोत के कारण समय पर प्रतिबंध को खत्म कर देता है। परिणामी बीजीय प्रणाली को फिर से हल किया जाता है, जिसके परिणामस्वरूप कम्प्यूटेशनल रूप से होता है कुशल कार्यान्वयन।

थीसिस के अंतिम योगदान में, हम दो-द्रव प्लाज्मा प्रवाह संतुलन पर विचार करते हैं कानून, जहां तीन आयाम मॉडल द्रव घटकों में टेन-मोमेंट समीकरण हैं। हम डेवलप करते हैं मॉडल के लिए दूसरा क्रम एंट्रॉपी स्थिर संख्यात्मक योजनाएं। इसे प्राप्त करने के लिए, हम पहले एक एन्ट्रापी रूढ़िवादी प्रवाह का निर्माण करें जो एंट्रॉपी के अतिरिक्त उपयुक्त रूप से संशोधित किया गया है प्रसार शब्द। फैलाना शब्द एन्ट्रापी स्केल

आइजेनवेक्टर का उपयोग करके बनाया गया है। इसके अलावा, स्रोत की शर्तों को स्पष्ट और निहित विवेक दोनों का उपयोग करके व्यवहार किया जाता है।

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List of Symbols

\mathbb{R}	Set of Real numbers
\mathbb{Z}	Set of Integers
\mathbb{N}	Set of Natural numbers
\mathbb{R}_+	Set of positive Real numbers
\mathbb{R}^d	Set of d-dimensional Real numbers
\subset	Subset
\in	Belongs to
\forall	For every
\implies	Implies
\otimes	Tensor outer product
Bold letters (<i>e.g.</i> \mathbf{u}, \mathbf{w})	Vector
∇	Gradient
∇	Divergence
$\nabla \times$	Curl
\mathbf{C}^1	Continuously differentiable function
\mathbf{C}^∞	Continuously differentiable function for all degrees
$\mathbf{C}_0^1(\mathbb{R}^d \times [0, \infty))$	\mathbf{C}^1 function with compact support in $\mathbb{R}^d \times [0, \infty)$
L_{loc}^∞	Space of locally bounded measurable functions
$(\cdot)^\top$	Transpose
$Sym(A)$	$\frac{1}{2}(A + A^\top)$