FAST ADAPTIVE MESHFREE WAVELET BASED METHODS FOR NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

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by

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Dedicated to

My Mother

Certificate

This is to certify that the thesis entitled **"Fast adaptive meshfree wavelet based methods for numerical solutions of partial differential equations and integral equations"** submitted by **"Ms. Kavita"** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for award of any degree or diploma.

New Delhi
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Abstract

Wavelet theory has been used for numerical solutions of partial differential equations (PDEs) since 1990s. There is a vast literature available on wavelet based methods for numerical solutions of PDEs. But the wavelet theory for numerical solutions of PDEs on general manifolds is still in its nascent stage. In this thesis, wavelet based fast adaptive and meshfree methods are developed which are easily extended to general manifolds (we have considered Euclidean domains, domain outside a butterfly in a square and the sphere). Diffusion wavelet and spectral graph wavelet (SGW) are used for this purpose. These two wavelets are already used in various areas of engineering, but to best of our knowledge not in the field of PDEs. We have made an attempt to exploit useful properties of these wavelets for numerical solutions of PDEs. Moreover, the diffusion wavelet based fast adaptive and meshfree method is used for numerical solutions of integral equations.

In order to have a better understanding of wavelet based methods for solving PDEs we started with Daubechies and spline-based wavelet. We developed a Matlab toolbox which contains the routines to compute the values of the scaling and wavelet functions ($\phi(x)$ and $\psi(x)$ respectively) and the derivatives of an arbitrary function (periodic or non periodic) using Daubechies and spline-based wavelet.

As our aim was to solve PDEs on general manifolds and Daubechies and spline wavelet based methods are limited to flat geometry, we shifted to diffusion wavelet and SGW.

Diffusion wavelet based fast adaptive and meshfree method is developed for numerical solutions of PDEs as well as of integral equations. This method uses finite difference (on the Euclidean domains) or radial basis functions (RBFs) (on the sphere) for space discretization and some suitable method (e.g. Crank Nicolson) for time discretization of a given PDE. Approximation formulae for Laplacian-Beltrami (∇^2) and gradient ($\vec{\nabla}$) operators are derived using RBFs on the unit sphere and unit cube, and the convergence of these approximations to ∇^2 and $\vec{\nabla}$ is derived and verified numerically. The diffusion wavelet is used for the following two purposes:

- For compression of the differential operators and hence for the fast computation of the powers of the matrices involved in the numerical solution of the PDEs.
- Diffusion wavelet coefficients are used as an indicator function to guide the refinement of the node arrangement and in this way this wavelet is used for adaptivity.

We have also proved that for the diffusion wavelet the compression error is bounded by the prescribed threshold ϵ as

$$\|f - f_{\geq\epsilon}\|_\infty \leq C\epsilon,$$

where C is determined by the function $f(x)$. This result ensures that when the wavelet is used for adaptivity, the error will remain bounded.

SGW based fast adaptive and meshfree method is developed for numerical solutions of PDEs on the sphere. This method uses RBFs for space discretization and some suitable method for time discretization of a given PDE. The SGW is used for the following purpose:

- SGW coefficients are used as an indicator function to guide the refinement of the node arrangement and in this way this wavelet is used for adaptivity.

Following are the characteristics of the above developed methods:

- Same matrix is used for the construction of the wavelet (both diffusion wavelet and SGW) as well as for approximations of the differential operators involved in PDEs.
- Methods are meshfree, and hence do not pose problems related to mesh generation.

The proposed methods are tested on a large number of test problems. Some of these test problems are elementary PDEs (Burger's equation) and some are more challenging problems (problem of computing a moving steep front on the sphere, the problem of pattern formation on the surface of the sphere using Turing equations). These numerical

tests show that the proposed methods can accurately capture the emergence of the localised patterns at all the scales and the node arrangement is accordingly adapted. The CPU time analysis of the methods reveals that the methods are efficient as compared to the traditional methods. The convergence of the proposed methods is derived and verified.

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