FOURTH ORDER FINITE DIFFERENCE METHODS FOR CERTAIN MILDLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

By

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A THESIS SUBMITTED TO THE
INDIAN INSTITUTE OF TECHNOLOGY, DELHI
FOR THE AWARD OF THE DEGREE OF
DOCTOR OF PHILOSOPHY

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JULY, 1989

DEDICATED
TO MY
PARENTS

CERTIFICATE

This is to certify that the thesis entitled "FOURTH ORDER FINITE DIFFERENCE METHODS FOR CERTAIN MILDLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS", which is being submitted by Ranjan Kumar Mohanty for the award of the degree, DOCTOR OF PHILOSOPHY in MATHEMATICS, to the Indian Institute of Technology, Delhi, is a bonafide record of research work done under our guidance and supervision.

The thesis has reached the standard fulfilling the requirements of the regulations relating to the degree. The results obtained in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

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ACKNOWLEDGEMENTS

It gives me great pleasure to express my profound sense of gratitude to my supervisors, Professor M.K. Jain and Prof. R.K. Jain, Department of Mathematics, Indian Institute of Technology, Delhi, for their valuable guidance, supervision, timely help and constant encouragement throughout the preparation of this thesis.

I express my sincere regards to Prof. M.M. Chawla, Head, Prof. O.P. Bhutani, former Head and Prof. S.R.K. Iyengar, Department of Mathematics, Indian Institute of Technology, Delhi, for their tremendous support and help provided to me during my research work.

I extend my thanks to the authorities of IIT Delhi, for providing me research facilities with financial support during the course of my research. I am also grateful to Prof.R.K.Arora, Head of the Computer Service Centre, IIT Delhi, for providing the necessary computer facilities.

I am greatly indebted to my father for giving me inspiration at every stage of my research work. I acknowledge his advice and inspiring suggestions with reverence. Personally, I owe a lot to my family members for their untiring patience and devotion.

Especially, I extend my thanks to Major Guru Prasad Mohanty, former Principal of B.J.B. College, Bhubaneswar, whom I shall remember for a long time for his constant encouragement and interest throughout my studies.

My sincere thanks are also due to all my fellow Research Scholars; especially Ms. Vanita Verma, Dr. (Mrs.) Pragya Jain, Mr. Rajive Kumar, Mr. P. Sampath Narayanan and Mr. M.V.Durga Prasad for their constant help and encouragements which had been a great source of strength and inspiration throughout my studies at IIT Delhi.

Finally, I would like to thank Mr. D.R. Joshi for his excellent, errorless and careful typing.

Mohanty

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SYNOPSIS

Most physical problems in science and engineering are described mathematically by partial differential equations. The expanding use of computers in studying such problems has made numerical methods an increasingly important tool for engineers and research scientists. Of the number of different numerical methods for solving partial differential equations, the most important are the finite difference methods (FDM). The basic merit of the FDM consists in their being universal and efficient, since they permit the determination of the numerical solution for a wide class of cases, and in application require but simple and quite realizable computation.

In this thesis, we have developed fourth order finite difference methods for numerical solution of certain nonlinear partial differential equations of the type-parabolic, elliptic and hyperbolic. The complete discussion of the thesis has been carried out in seven chapters. This is followed by a bibliograph useful for the development and application of the methods discussed in this thesis. We now give briefly the main results obtained in each chapter.

CHAPTER I: NUMERICAL METHODS FOR SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

This chapter presents introduction, a brief review of the numerical methods and the summary of the work done in the present thesis.

CHAPTER II: A HIGH ORDER FINITE DIFFERENCE METHOD FOR 1-D PARABOLIC EQUATIONS WITH NONLINEAR FIRST DERIVATIVE TERMS

A two level implicit 6-point difference scheme of $O(k^2+kh^2+h^4)$ has been derived for solving the 1-D parabolic equation

$$u_{xx} = f(x,t,u,u_x,u_t)$$

subject to Dirichlet conditions. The difference method when applied to the linear diffusion-convection problem becomes of $O(k^2+h^4)$. It is also shown that the difference scheme is unconditionally stable. The numerical results have been compared with those obtained by using the $O(k^2+h^2)$ method and the exact solution values to illustrate the efficiency of this method.

CHAPTER III: A HIGH ORDER DIFFERENCE METHOD FOR 2-D PARABOLIC EQUATIONS WITH NONLINEAR FIRST DERIVATIVE TERMS

A two level implicit 18-point scheme of order $(k^2+kh^2+h^4)$ and unconditionally stable has been derived for solving the 2-D parabolic equation

$$\nu_1 u_{xx} + \nu_2 u_{yy} = f(x,y,t,u,u_x, u_y, u_t)$$

subject to Dirichlet conditions.
The numerical results of the linear diffusion-convection equation with constant coefficients have been obtained by using the splitting operator technique whereas for the problem with nonlinear terms the numerical results have been determined by

using the composite difference scheme. Numerical results have been compared with those obtained by using the $O(k^2+h^2)$ method and the exact solution values.

CHAPTER IV: A HIGH ORDER FINITE DIFFERENCE METHOD FOR 3-D PARABOLIC EQUATIONS WITH NONLINEAR FIRST DERIVATIVE TERMS

A two level implicit 38-point difference scheme of $O(k^2+kh^2+h^4)$ and unconditionally stable has been derived for solving the 3-D parabolic equation

$$\nu_1 u_{xx} + \nu_2 u_{yy} + \nu_3 u_{zz} = f(x,y,z,t,u,u_x,u_y,u_z,u_t)$$

subject to Dirichlet conditions. The numerical results have been obtained to illustrate the efficiency of the method.

CHAPTER V:* FOURTH ORDER DIFFERENCE METHODS FOR 2-D AND 3-D MILDLY NONLINEAR ELLIPTIC EQUATIONS

The convergence of $O(h^4)$ methods for 2-D and 3-D elliptic equations has been discussed. The Jacobi iteration method has

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*The results reported in this chapter for 2-D elliptic equations have been accepted for publication. "A Fourth-Order Difference Method for Two-Dimensional Elliptic Equations with Nonlinear First Derivative Terms" Numerical Methods for Partial Differential Equations, Forthcoming, (1989).

also been studied and found to converge for all cell Reynold numbers. Numerical results have been compared with those obtaine by using the $0(h^2)$ methods and the exact solution values to prove that the fourth order methods give accurate results.

CHAPTER VI: FOURTH ORDER DIFFERENCE METHODS FOR 1-D AND 2-D MILDLY NONLINEAR HYPERBOLIC EQUATIONS

Three level implicit schemes of $0(k^4+h^4)$ have been derived for 1-D and 2-D mildly nonlinear hyperbolic equations of the form

$$u_{tt} = Au_{xx} + f(x,t,u,u_x,u_t), \quad A > 0$$

and

$$u_{tt} = Au_{xx} + Bu_{yy} + f(x,y,t,u,u_x,u_y,u_t)$$

where $A, B > 0$, with appropriate initial and boundary conditions. The difference schemes are conditionally stable. The linear problems in 2-D have been solved using splitting operator technique whereas the nonlinear equations are solved using composite scheme.

Numerical results for both 1-D and 2-D problems are given and compared with those obtained using the $0(k^2+h^2)$ methods and the exact solution values.

CHAPTER VII: A FOURTH ORDER FINITE DIFFERENCE METHOD FOR 3-D SECOND ORDER MILDLY NONLINEAR HYPERBOLIC EQUATIONS

A three level implicit 33-point difference scheme of $0(k^4+h^4)$ has been developed for solving the 3-D second order mildly nonlinear hyperbolic equation

$$u_{tt} = Au_{xx} + Bu_{yy} + Cu_{zz} + f(x,y,z,t,u,u_x,u_y,u_z,u_t)$$

where A,B,C > 0, with appropriate initial and boundary conditions The stability analysis has been derived by applying the differenc method to the test problem

$$u_{tt} = Au_{xx} + Bu_{yy} + Cu_{zz} + \alpha u_x + \beta u_y + \gamma u_z$$

where α, β and γ are constants.

The numerical results of two linear and one nonlinear problems are given.

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