

FOURTH ORDER FINITE DIFFERENCE METHODS FOR CERTAIN MILDLY NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

By

RANJAN KUMAR MOHANTY

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Department of Mathematics
INDIAN INSTITUTE OF TECHNOLOGY, DELHI
HAUZ KHAS, NEW DELHI-110016
INDIA
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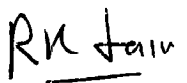
CERTIFICATE

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*The thesis has reached the standard fulfilling the require-
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University or Institute for the award of any degree or diploma.*


(M.K. JAIN)

*Department of Mathematics
Indian Institute of Technology, Delhi
Hauz Khas, New Delhi-110016.*


(R.K. JAIN)

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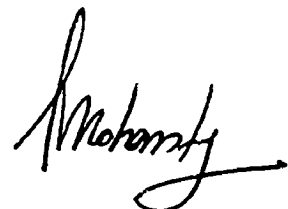
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SYNOPSIS

Most physical problems in science and engineering are described mathematically by partial differential equations. The expanding use of computers in studying such problems has made numerical methods an increasingly important tool for engineers and research scientists. Of the number of different numerical methods for solving partial differential equations, the most important are the finite difference methods (FDM). The basic merit of the FDM consists in their being universal and efficient, since they permit the determination of the numerical solution for a wide class of cases, and in application require but simple and quite realizable computation.

In this thesis, we have developed fourth order finite difference methods for numerical solution of certain nonlinear partial differential equations of the type-parabolic, elliptic and hyperbolic. The complete discussion of the thesis has been carried out in seven chapters. This is followed by a bibliography useful for the development and application of the methods discussed in this thesis. We now give briefly the main results obtained in each chapter.

CHAPTER I: NUMERICAL METHODS FOR SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

This chapter presents introduction, a brief review of the numerical methods and the summary of the work done in the present thesis.

CHAPTER II: A HIGH ORDER FINITE DIFFERENCE METHOD FOR
1-D PARABOLIC EQUATIONS WITH NONLINEAR
FIRST DERIVATIVE TERMS

A two level implicit 6-point difference scheme of $O(k^2 + kh^2 + h^4)$ has been derived for solving the 1-D parabolic equation

$$u_{xx} = f(x, t, u, u_x, u_t)$$

subject to Dirichlet conditions. The difference method when applied to the linear diffusion-convection problem becomes of $O(k^2 + h^4)$. It is also shown that the difference scheme is unconditionally stable. The numerical results have been compared with those obtained by using the $O(k^2 + h^2)$ method and the exact solution values to illustrate the efficiency of this method.

CHAPTER III: A HIGH ORDER DIFFERENCE METHOD FOR 2-D PARABOLIC
EQUATIONS WITH NONLINEAR FIRST DERIVATIVE TERMS

A two level implicit 18-point scheme of order $(k^2 + kh^2 + h^4)$ and unconditionally stable has been derived for solving the 2-D parabolic equation

$$v_1 u_{xx} + v_2 u_{yy} = f(x, y, t, u, u_x, u_y, u_t)$$

subject to Dirichlet conditions.

The numerical results of the linear diffusion-convection equation with constant coefficients have been obtained by using the splitting operator technique whereas for the problem with nonlinear terms the numerical results have been determined by

using the composite difference scheme. Numerical results have been compared with those obtained by using the $O(k^2+h^2)$ method and the exact solution values.

CHAPTER IV: A HIGH ORDER FINITE DIFFERENCE METHOD FOR 3-D PARABOLIC EQUATIONS WITH NONLINEAR FIRST DERIVATIVE TERMS

A two level implicit 38-point difference scheme of $O(k^2+kh^2+h^4)$ and unconditionally stable has been derived for solving the 3-D parabolic equation

$$v_1 u_{xx} + v_2 u_{yy} + v_3 u_{zz} = f(x, y, z, t, u, u_x, u_y, u_z, u_t)$$

subject to Dirichlet conditions. The numerical results have been obtained to illustrate the efficiency of the method.

CHAPTER V: ^{*}FOURTH ORDER DIFFERENCE METHODS FOR 2-D AND 3-D MILDLY NONLINEAR ELLIPTIC EQUATIONS

The convergence of $O(h^4)$ methods for 2-D and 3-D elliptic equations has been discussed. The Jacobi iteration method has

*The results reported in this chapter for 2-D elliptic equations have been accepted for publication. "A Fourth-Order Difference Method for Two-Dimensional Elliptic Equations with Nonlinear First Derivative Terms" Numerical Methods for Partial Differential Equations, Forthcoming, (1989).

also been studied and found to converge for all cell Reynold numbers. Numerical results have been compared with those obtained by using the $O(h^2)$ methods and the exact solution values to prove that the fourth order methods give accurate results.

CHAPTER VI: FOURTH ORDER DIFFERENCE METHODS FOR 1-D AND 2-D MILDLY NONLINEAR HYPERBOLIC EQUATIONS

Three level implicit schemes of $O(k^4+h^4)$ have been derived for 1-D and 2-D mildly nonlinear hyperbolic equations of the form

$$u_{tt} = Au_{xx} + f(x,t,u,u_x,u_t), \quad A > 0$$

and

$$u_{tt} = Au_{xx} + Bu_{yy} + f(x,y,t,u,u_x,u_y,u_t)$$

where $A, B > 0$, with appropriate initial and boundary conditions. The difference schemes are conditionally stable. The linear problems in 2-D have been solved using splitting operator technique whereas the nonlinear equations are solved using composite scheme.

Numerical results for both 1-D and 2-D problems are given and compared with those obtained using the $O(k^2+h^2)$ methods and the exact solution values.

CHAPTER VII: A FOURTH ORDER FINITE DIFFERENCE METHOD FOR 3-D SECOND ORDER MILDLY NONLINEAR HYPERBOLIC EQUATIONS

A three level implicit 33-point difference scheme of $O(k^4+h^4)$ has been developed for solving the 3-D second order mildly nonlinear hyperbolic equation

$$u_{tt} = Au_{xx} + Bu_{yy} + Cu_{zz} + f(x, y, z, t, u, u_x, u_y, u_z, u_t)$$

where $A, B, C > 0$, with appropriate initial and boundary conditions
The stability analysis has been derived by applying the difference method to the test problem

$$u_{tt} = Au_{xx} + Bu_{yy} + Cu_{zz} + \alpha u_x + \beta u_y + \gamma u_z$$

where α , β and γ are constants.

The numerical results of two linear and one nonlinear problems are given.

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