Overlapping Schwarz Domain Decomposition Methods for Singularly Perturbed Reaction-Diffusion Problems

By

Sunil Kumar

Department of Mathematics

Submitted in fulfillment of the requirements of the degree of Doctor of Philosophy

to the

Indian Institute of Technology Delhi

August 2011

Certificate

This is to certify that the thesis entitled **Overlapping Schwarz Domain Decomposition Methods for Singularly Perturbed Reaction-Diffusion Problems** submitted by **Mr Sunil Kumar** to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by him under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
August 2011

Dr S. Chandra Sekhara Rao
Associate Professor
Department of Mathematics
Indian Institute of Technology Delhi
New Delhi-110 016

Acknowledgements

The journey of completing a PhD thesis is culminations of never-ending efforts and in pursue of the same, one requires the involvement of multiple individuals. I take this opportunity to express my thanks to all those who have guided, encouraged, remained patient and showed faith in me to conclude my academic pilgrimage.

First and foremost, I would like to express my deepest regards to my thesis supervisor Dr S. Chandra Sekhara Rao. I am extremely fortunate to have a supervisor who gave me the freedom to explore on my own, and at the same time the guidance to recover whenever my steps faltered. At every crucial juncture, I was amazed at Dr Rao's innate ability to anticipate my concerns and to skillfully guide me to the right direction. His enthusiasm, knowledge, support and patience throughout the program kept me on track and encouraged me when the going got tougher.

I am also deeply grateful to Prof. R. K. Sharma, for the kind cooperation he extended to me in the capacities: as a Chairperson DRC(Department Research Committee), and as a Head, Mathematics Department. I would like to give my special thanks to my SRC (Student Research Committee) members Dr K. Sreenadh and Prof. A. D. Rao, for their valuable time and suggestions. I am greatly indebted to all faculty members of Department of Mathematics IIT Delhi, for their co-operation and support.

Many thanks to IIT Delhi authorities for providing me the research fellowship and necessary facilities all through the PhD program. I would also like to express my sincere appreciation to NBHM and DST for providing me the financial support to attend the BAIL conference (2010) in Spain. This gave me a tremendous opportunity to network with extremely knowledgeable professionals and further helped me keeping abreast with the latest technological advances.

The contributions made by my friend Dr Mukesh Kumar in overcoming the initial hurdles of research orientation are highly appreciated. I am highly grateful to him for being there, when I needed him the most. The support and cooperation extended by my hostel caretaker Mr Abhilash Poonia are mentioned here with many thanks.

I am immeasurably grateful to my family who gently offered counseling and gave unconditional support at each and every point of my academic journey. It was the patience and silent sacrifice of my mother, Munesh, which led me to complete this sojourn. My achievements are outcome of her dedication and her belief in my potentials. It is her blessings that made me reach this milestone. I am thankful to my uncles Deshpal Singh, K. P. Singh, and aunt Bala who extended their valuable support and continuous encouragement all throughout my life. Thanks to my brothers (Sunny, Neeraj, Amit and Vikas) and my sisters (Nisha, Anjali and Amrita) for their love and kindness through this long process. Last but not the least, I acknowledge my deep gratitude to the memory of my beloved father, who would have been proud of the biggest accomplishment of his son and I thank him for his everlasting blessings which he is still bestowing on me.

I also like to express a heartfelt thank to all my friends Dina, Sunil, Alok, Arun, Sachin, Sarvesh, Anikendra, and Balchand who generously shared time at different stages of my work, and the enjoyable moments spent with them will always be memorable. Many thanks to my senior colleagues Dr Pramod and Dr R. K. Pandey for their company and the laughs we shared along the way.

I am deeply indebted to IIT Delhi cricket team for giving me cherished memories including gold medals at IIT Bombay and at IIT Kanpur. A special mentioning about few characters especially of Roushan, Sandeep, Arvindar, Randhava, Mittal and coach Abhinav, whose involvement during my IIT Delhi stint have tremendous impact on my social thinking and belief.

Most importantly, I thank almighty god for his blessings in helping me reach this landmark.

New Delhi *Sunil Kumar*

Abstract

This thesis is concerned with the design and analysis of overlapping Schwarz domain decomposition methods for singularly perturbed reaction-diffusion problems. These problems arise in various application fields, such as fluid dynamics, gas dynamics, elasticity, control theory and chemical kinetics. In general the solutions to these problems exhibit layers behavior. Classical numerical approaches for these problems are inadequate as they require prohibitively large numbers of mesh points to produce satisfactory approximations. It is of both theoretical and practical interest to construct special numerical methods for these problems where accuracy is guaranteed independent of the size of the perturbation parameter.

A high order overlapping Schwarz domain decomposition method is designed for solving singularly perturbed linear reaction-diffusion problems. The numerical approximations obtained from this method are proved to be almost fourth order uniformly convergent. Furthermore, we address the accelerated convergence of the algorithm for small ε. To be more precise, we prove that, when ε is small, only one iteration is required to achieve almost fourth order uniform convergence. We then extend the method to systems of $M(\geq 2)$ singularly perturbed linear reaction-diffusion problems.

We next propose an overlapping Schwarz domain decomposition method for solving singularly perturbed semilinear reaction-diffusion problems. We prove that the method is almost fourth order uniformly convergent, and that for small ε, we only require one iteration. Further, this method is extended to systems of $M(\geq 2)$ singularly perturbed semilinear reaction-diffusion problems.

Two overlapping Schwarz domain decomposition methods are designed for solving time dependent singularly perturbed linear reaction-diffusion problems. The first method gives uniform numerical approximations of first order in time and almost second order in space. The second method gives uniform numerical approximations of first order in time and almost fourth order in space. Furthermore, it has been proved that, when ε is small, one iteration is sufficient to achieve the expected accuracy. These two methods are extended to systems of $M(\geq 2)$ time dependent singularly perturbed linear reaction-diffusion problems.

Numerical experiments support the theoretical results and demonstrate the effectiveness of all the designed methods.

Contents

Notation

ε	singular perturbation parameter
k	iteration number
$N,\ N_t$	mesh discretization parameters
O	Landau symbol
Ω	given space variable domain
σ	subdomain parameter
$\Omega_\ell, \Omega_m, \Omega_r$	subdomains of Ω
$\overline{\Omega}^N$	$(\overline{\Omega}_\ell^N \setminus \overline{\Omega}_m) \cup \overline{\Omega}_m^N \cup (\overline{\Omega}_r^N \setminus \overline{\Omega}_m)$
$Q = \Omega \times (0, T]$	given domain for non-stationary problems
Q_ℓ, Q_m, Q_r	subdomains of Q
$\overline{Q}^{N,N_t}$	$(\overline{Q}_\ell^{N,N_t} \setminus \overline{Q}_m) \cup \overline{Q}_m^{N,N_t} \cup (\overline{Q}_r^{N,N_t} \setminus \overline{Q}_m)$
$C^s(\Omega),\ C^{s,n}(Q)$	function spaces
$\boldsymbol{v}$	$(v_1, \ldots, v_M)^T$
$\|\|v\|\|_D$	$\max_{x \in D} \|v(x)\|,\ D \subset \Omega$
$\|\|\boldsymbol{v}\|\|_D$	$\max\{\|\|v_1\|\|_D, \ldots, \|\|v_M\|\|_D\}$
$\|\|.\|\|$	when domain is obvious, or of no particular significance
v_i	$v(x_i)$
$v_{p;i}$	$v_p(x_i)$
$\boldsymbol{v}_i$	$\boldsymbol{v}(x_i) = (v_{1;i}, \ldots, v_{M;i})^T$

$\boldsymbol{v}_{p;i}$	$\boldsymbol{v}_p(x_i) = (v_{p,1;i}, \dots, v_{p,M;i})^T$
$\boldsymbol{v} \le \boldsymbol{w}$	if $v_n \le w_n,\ 1 \le n \le M$
$\lvert\boldsymbol{v}\rvert$	$(\lvert v_1\rvert, \dots, \lvert v_M\rvert)^T$
$\lVert\boldsymbol{v}\rVert_\star$	$\max\limits_{1\le n\le M} \lvert v_n\rvert$
$\lvert\lvert v\rvert\rvert_{\overline{\Omega}^N}$	$\max\limits_{x_i\in\overline{\Omega}^N} \lvert v(x_i)\rvert$
$\lvert\lvert \boldsymbol{v}\rvert\rvert_{\overline{\Omega}^N}$	$\max\{\lvert\lvert v_1\rvert\rvert_{\overline{\Omega}^N}, \dots, \lvert\lvert v_M\rvert\rvert_{\overline{\Omega}^N}\}$
$\lvert\lvert w\rvert\rvert_{\overline{Q}^{N,N_t}}$	$\max\limits_{(x_i,t_j)\in\overline{Q}^{N,N_t}} \lvert w(x_i,t_j)\rvert$
$\lvert\lvert \boldsymbol{w}\rvert\rvert_{\overline{Q}^{N,N_t}}$	$\max\{\lvert\lvert w_1\rvert\rvert_{\overline{Q}^{N,N_t}}, \dots, \lvert\lvert w_M\rvert\rvert_{\overline{Q}^{N,N_t}}\}$
$C,\ C_s,\ s = 0, 1, \dots$	generic positive constants, independent of ε, N, k and N_t
$\boldsymbol{C},\ \boldsymbol{C}_s, s = 0, 1, \dots$	generic positive constant vectors

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