

Overlapping Schwarz Domain Decomposition Methods for Singularly Perturbed Reaction-Diffusion Problems

By

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Certificate

This is to certify that the thesis entitled **Overlapping Schwarz Domain Decomposition Methods for Singularly Perturbed Reaction-Diffusion Problems** submitted by **Mr Sunil Kumar** to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by him under my supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
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Abstract

This thesis is concerned with the design and analysis of overlapping Schwarz domain decomposition methods for singularly perturbed reaction-diffusion problems. These problems arise in various application fields, such as fluid dynamics, gas dynamics, elasticity, control theory and chemical kinetics. In general the solutions to these problems exhibit layers behavior. Classical numerical approaches for these problems are inadequate as they require prohibitively large numbers of mesh points to produce satisfactory approximations. It is of both theoretical and practical interest to construct special numerical methods for these problems where accuracy is guaranteed independent of the size of the perturbation parameter.

A high order overlapping Schwarz domain decomposition method is designed for solving singularly perturbed linear reaction-diffusion problems. The numerical approximations obtained from this method are proved to be almost fourth order uniformly convergent. Furthermore, we address the accelerated convergence of the algorithm for small ε . To be more precise, we prove that, when ε is small, only one iteration is required to achieve almost fourth order uniform convergence. We then extend the method to systems of $M(\geq 2)$ singularly perturbed linear reaction-diffusion problems.

We next propose an overlapping Schwarz domain decomposition method for solving singularly perturbed semilinear reaction-diffusion problems. We prove that the method is almost fourth order uniformly convergent, and that for small ε , we only

require one iteration. Further, this method is extended to systems of $M(\geq 2)$ singularly perturbed semilinear reaction-diffusion problems.

Two overlapping Schwarz domain decomposition methods are designed for solving time dependent singularly perturbed linear reaction-diffusion problems. The first method gives uniform numerical approximations of first order in time and almost second order in space. The second method gives uniform numerical approximations of first order in time and almost fourth order in space. Furthermore, it has been proved that, when ε is small, one iteration is sufficient to achieve the expected accuracy. These two methods are extended to systems of $M(\geq 2)$ time dependent singularly perturbed linear reaction-diffusion problems.

Numerical experiments support the theoretical results and demonstrate the effectiveness of all the designed methods.

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Notation

ε	singular perturbation parameter
k	iteration number
N, N_t	mesh discretization parameters
O	Landau symbol
Ω	given space variable domain
σ	subdomain parameter
$\Omega_\ell, \Omega_m, \Omega_r$	subdomains of Ω
$\bar{\Omega}^N$	$(\bar{\Omega}_\ell^N \setminus \bar{\Omega}_m^N) \cup \bar{\Omega}_m^N \cup (\bar{\Omega}_r^N \setminus \bar{\Omega}_m^N)$
$Q = \Omega \times (0, T]$	given domain for non-stationary problems
Q_ℓ, Q_m, Q_r	subdomains of Q
\bar{Q}^{N, N_t}	$(\bar{Q}_\ell^{N, N_t} \setminus \bar{Q}_m) \cup \bar{Q}_m^{N, N_t} \cup (\bar{Q}_r^{N, N_t} \setminus \bar{Q}_m)$
$C^s(\Omega), C^{s,n}(Q)$	function spaces
\mathbf{v}	$(v_1, \dots, v_M)^T$
$\ v\ _D$	$\max_{x \in D} v(x) , D \subset \Omega$
$\ \mathbf{v}\ _D$	$\max\{\ v_1\ _D, \dots, \ v_M\ _D\}$
$\ \cdot\ $	when domain is obvious, or of no particular significance
v_i	$v(x_i)$
$v_{p;i}$	$v_p(x_i)$
\mathbf{v}_i	$\mathbf{v}(x_i) = (v_{1;i}, \dots, v_{M;i})^T$

$\mathbf{v}_{p;i}$	$\mathbf{v}_p(x_i) = (v_{p,1;i}, \dots, v_{p,M;i})^T$
$\mathbf{v} \leq \mathbf{w}$	if $v_n \leq w_n, 1 \leq n \leq M$
$ \mathbf{v} $	$(v_1 , \dots, v_M)^T$
$\ \mathbf{v}\ _*$	$\max_{1 \leq n \leq M} v_n $
$\ v\ _{\bar{\Omega}^N}$	$\max_{x_i \in \bar{\Omega}^N} v(x_i) $
$\ \mathbf{v}\ _{\bar{\Omega}^N}$	$\max\{\ v_1\ _{\bar{\Omega}^N}, \dots, \ v_M\ _{\bar{\Omega}^N}\}$
$\ w\ _{\bar{Q}^{N,N_t}}$	$\max_{(x_i, t_j) \in \bar{Q}^{N,N_t}} w(x_i, t_j) $
$\ \mathbf{w}\ _{\bar{Q}^{N,N_t}}$	$\max\{\ w_1\ _{\bar{Q}^{N,N_t}}, \dots, \ w_M\ _{\bar{Q}^{N,N_t}}\}$
$C, C_s, s = 0, 1, \dots$	generic positive constants, independent of ε, N, k and N_t
$\mathbf{C}, \mathbf{C}_s, s = 0, 1, \dots$	generic positive constant vectors

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