ON THE UNIT GROUP OF CERTAIN FINITE GROUP ALGEBRAS

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by

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Dedicated to

My Family

Certificate

This is to certify that the thesis entitled **"ON THE UNIT GROUP OF CERTAIN FINITE GROUP ALGEBRAS"** submitted by **Mr. Yogesh Kumar** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

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Abstract

Let G be a finite group, $\mathbb{F}_q$ a finite field of q elements and $\mathbb{F}_qG$ be the group algebra of G over $\mathbb{F}_q$. This thesis is a study of unit groups of certain finite group algebras. For this, we compute the dimension of the Jacobson radical $J(\mathbb{F}_qG)$ and the Wedderburn decomposition of $\frac{\mathbb{F}_qG}{J(\mathbb{F}_qG)}$. In particular, we obtain Wedderburn decomposition of $\frac{\mathbb{F}_qS_5}{J(\mathbb{F}_qS_5)}$, where S_5 is the symmetric group of degree 5, and $\mathbb{F}_q$ is any finite field. We determined the unit group $\mathcal{U}(\mathbb{F}_qGL_2(\mathbb{Z}_3))$ of the group algebra $\mathbb{F}_qGL_2(\mathbb{Z}_3)$ where $GL_2(\mathbb{Z}_3)$ denotes the multiplicative group of 2×2 matrices over the ring of integers modulo 3. Further we discuss the unit group of the group algebras for groups $\mathbb{Z}_9 \rtimes \mathbb{Z}_3$ and $(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$ where $\mathbb{Z}_9$ and $\mathbb{Z}_3$ denotes the multiplicative cyclic groups of order 9 and 3 respectively, over any finite field. Finally, we determined the unit group $\mathcal{U}(\mathbb{F}_qD_{40})$ of the group algebra $\mathbb{F}_qD_{40}$, of the dihedral group D_{40} of order 40, over any finite field $\mathbb{F}_q$.

सार

माना G एक सीमित समूह है, F_q, q तत्वों का एक सीमित छेत्र और FqG, G का Fq पर ग्रुप अलजेब्रा है। यह थीसिस कुछ सीमित ग्रुप अलजेब्रा के यूनिट समूहों का अध्ययन है । इसके लिए, हम जैकबसन रेडिकल J(FqG) की गणना करते हैं और FqG की वंडरबर्न अपघटन प्राप्त करते हैं । विशेष रूप से, हम FqS_5/J $(Fq\ S_5)$ के वेडरबर्न अपघटन को प्राप्त करते हैं, जहां S_5 डिग्री 5 का सममित समूह है, और Fq कोई भी सीमित छेत्र है। हमने ग्रुप अलजेब्रा $FqGL_2(Z_3)$ की इकाई समूह $U(FqGL_2(Z_3))$ को निर्धारित किया, जहां $GL_2(Z_3)$, 2×2 की मैट्रिक्स रिंग को दर्शाता हैं जो इंटीग्रल मॉडलों 3 पर है । आगे हम ग्रुप अलजेब्रा के यूनिट समूह के लिए चर्चा करते हैं, जहां हमारा समूह $Z9\rtimes Z3$ और $(Z3 \times Z3) \rtimes Z3$ है । अंत में, हमने इकाई समूह $U(FqD_{40})$ की गणना कि , जहा हमारा ग्रुप अलजेब्रा $F_q\ D_{40}$ है । D_{40} फोर्टी आर्डर का दिहेड्रल ग्रुप है ।

Contents

List of Tables

List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	the set of natural numbers with zero
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
p	a prime
q	a prime power
$x \mid y$	x divides y
$x \nmid y$	x does not divide y
$\lvert A \rvert$	cardinality of the set A
$x \equiv y \pmod{m}$	x is congruent to y modulo m
$\gcd(a_1, a_2, \ldots, a_n)$	the greatest common divisor of $a_1, \ldots, a_n$
$\phi(m)$	Euler's phi-function of m
$\mathbb{Z}_n$	set of integers modulo n
$U(\mathbb{Z}_n)$	the multiplicative group of integers modulo n
$\langle a \rangle$	the cyclic group generated by a
$\langle a, b, c, \cdots \mid X, Y, Z, \cdots \rangle$	presentation of the group generated by $a, b, c, \cdots$ subject to the relations $X = 1, Y = 1, Z = 1, \cdots$

$\mathbb{F}$	any field
$\mathbb{F}_q$, $GF(q)$	finite field with q elements
$\mathbb{F}_q^*$	the multiplicative group of nonzero elements of $\mathbb{F}_q$
S_n	symmetric group on n symbols
A_n	alternating group on n symbols
$char(R)$	characteristic of R
$R \cong S$	R is isomorphic to S
$\mathcal{U}(R)$	unit group of the ring R
C_n	cyclic group of order n
$\rtimes$	semi direct product
$\oplus$	direct sum
$\otimes$	tensor product
$J(R)$	the Jacobson radical of the ring R
$\|G : N\|$	index of a normal subgroup N in group G
$dim_{\mathbb{F}}(V)$	dimension of a vector space V over $\mathbb{F}$
G'	the commutator subgroup of G
$SL_n(\mathbb{F})$	the special linear group of $n \times n$ matrices over the field $\mathbb{F}$
$GL_n(\mathbb{F})$	the general linear group of $n \times n$ matrices over the field $\mathbb{F}$
$N_G(X)$	the normalizer of X in G
$C_G(X)$	the centralizer of X in G
$\mathbb{F}G$	group algebra of G over $\mathbb{F}$
$\Delta(G)$	the augmentation ideal of $\mathbb{F}G$
G^n	the set $\{g^n \mid g \in G\}$
$\mathbb{M}_n(R)$	the ring of all $n \times n$ matrices over R
$t(G)$	the nilpotency index of $J(\mathbb{F}G)$
D_{2n}	the dihedral group of order $2n$
$V(\mathbb{F}G)$	the set of augmentation 1 invertible elements of $\mathbb{F}G$
$V_*(\mathbb{F}G)$	the set of unitary units of $\mathbb{F}G$ with canonical involution $*$

S^{μ}	the specht module corresponding to partition μ
D^{μ}	$\frac{S^{\mu}}{S^{\mu} \cap S^{\mu\perp}}$ for a p-regular partition μ
O_p	the maximal normal p-subgroup
$O_{p'}$	the maximal normal p'-subgroup
$o(x)$	the order of element x
x^g	group action of element g on x