MECHANICS OF HIGHLY FLEXIBLE MANIPULATORS

SREEJATH. S

DEPARTMENT OF MECHANICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2025

MECHANICS OF HIGHLY FLEXIBLE MANIPULATORS

by

SREEJATH. S

DEPARTMENT OF MECHANICAL ENGINEERING

submitted

in fulfilment of the requirements of the degree of doctor of philosophy

to the

INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY 2025

CERTIFICATE

This is to certify that the thesis titled **Mechanics of Highly Flexible Manipulators**, submitted by **Sreejath. S**, to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Prof. S. K. Saha
Professor
Dept. of Mechanical Engineering
IIT-Delhi, 110016

Prof. S. P. Singh
Professor
Dept. of Mechanical Engineering
IIT-Delhi, 110016

Date: 10 July 2025

Place: New Delhi

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisors, Prof. S. K. Saha, and Prof. S. P. Singh, for their invaluable guidance, support, and encouragement throughout my journey. Their insights and expertise were instrumental in shaping my research. This thesis would not have been possible without their constant motivation, especially during challenging times.

I am deeply thankful to my Student Research Committee members Prof. J. K. Dutt, Prof. N. V. Datla, and Prof. Sitikanta Roy for their constructive feedback and valuable suggestions, which greatly enhanced the quality of my work. I also thank Prof. Ajeet Kumar, whose online classes significantly helped me grasp key concepts and clear my doubts.

I am also extremely thankful to Prof. Ashish Suri, a neurosurgeon at AIIMS Delhi, for providing the medical inputs, and the laboratory facilities necessary for developing the surgical manipulator as part of an intramural project between Indian Institute of Technology Delhi (IITD) and All India Institute of Medical Science Delhi (AIIMS-D). I am also grateful to Dr. Ramandeep Singh and Mr. Rajdeep Singh for their invaluable role in design, fabrication, and testing of the prototype of the surgical manipulator

For half of my Ph.D. journey, I had the privilege of sharing the research scholar room with Dr. Vishnu Sukumar, Dr. Suraj Bhat, Dr. Alinjar Dan, and Dr. Sasanka Sekhar Sinha. Our discussions on our respective works were both enlightening and motivating, and their company was a pleasure. I also extend my heartfelt thanks to my seniors, Dr. Paramanand

Nandihal, Dr. Anil Sharma, and Mr. Dinesh Kochar for their valuable advice and suggestions that significantly contributed to the direction of my research. I am grateful to my colleague, Mr. Sandeep Kumar, for his assistance in conducting experiments and taking part in discussion. A special mention to Mr. Surya Pratap Rathor, whom with I had deep theoretical discussions and got valuable suggestions. My second phase of the journey was witnessed by my fellow lab mates in the Vibration Research Lab. I sincerely thank Prof. Kshitij Gupta, Prof. S. P. Singh, Prof. Ashish K. Darpe, Mr. Madhu, and Mr. Abhishek Rana for providing me with the facilities to work there. My lab mates also deserve thanks for creating a supportive environment. I thank Ms. Bhashi for helping me with modelling and experiments. I would like to thank Prof. Enrico Bertolazzi of the Department of Industrial Engineering at the University of Trento, Italy, for his contribution to this work. His public libraries of clothoid interpolation proved to be very helpful.

The journey wouldn't be complete without acknowledging the unwavering support of my friends. Thanks to my close friends for including me in regular trips which provided much-needed breaks. I am thankful to IITD for providing me with nice playgrounds, helping me stay fit. Thanks to all my football and frisbee teammates for the countless good times playing the sport. The memorable support came from my soulmate and future life partner, Dr. Charutha V. V. Her selfless and sincere support, both financial and emotional, during the critical stages of my Ph.D. was invaluable. I thank her parents for patiently waiting for me to graduate. I am also grateful to Mr. Hridin Pradeep and Mr. Binoy. B, who provided immense support and brotherly care whenever I needed it.

Finally, I express my deepest gratitude to my family for their unwavering belief in me. They showed immense patience by relieving me from major family responsibilities until my thesis submission. I am especially thankful to my mother, Mrs. Sreedevi S., and father, Mr. Sivaj S., for providing me with opportunities for my studies and their unwavering emotional and financial support. Words cannot express my gratitude to my twin brother, Mr. Sreejith S, for helping and immensely supporting my parents during my absence. I also thank my sister, Mrs. Sreeja S., and brother-in-law, Mr. Suresh S., for their constant care and love. Finally, thanks to my nephew, master Adithyadev S, and baby Uthara S, for waiting patiently for me during my vacations. I would like to thank all those who have supported me in any way during the completion of my thesis.

S. Sreejath

ABSTRACT

Highly flexible manipulators offer unique advantages in various applications due to their dexterity, lightweight design, and ability to navigate complex environments. However, their inherent flexibility presents significant challenges in terms of modeling, control, and accurate positioning. This thesis investigates the mechanics of highly flexible manipulators to address these challenges and enable their effective utilization in diverse fields. Understanding the mechanical behavior of highly flexible manipulators is crucial for their successful design, control, and operation. Due to their significant deflection and complex dynamic characteristics, traditional rigid-body manipulator models fail to capture their behavior accurately. This thesis explores the mechanics of highly flexible manipulators, developing models and analysis techniques to predict their motion and design them for optimal performance.

First portion of the thesis describes the modelling approaches in highly deformable structures. Generally, conventional approaches such as Euler-Bernoulli beam models are insufficient to model highly deformable structures due to their inherent assumptions of small rotation. Hence more sophisticated methods are required which relaxes the assumptions. Geometrically-exact beam theories are of such category. Mainly, continuous, and discrete methods are employed to model highly deformable bodies. Continuous methods treat deformable bodies as continuum and derive the governing partial differential equations that represent the dynamic behaviours. One of such methods is Cosserat rod modelling, which is very popular in modelling continuum manipulators. On the other hand, discrete methods such as the Finite Element Method (FEM) discretize the continuum structures into a finite number of elements to obtain a numerical solution for the manipulator's mechani-

cal response. Cosserat rod model offers a robust framework to model flexible manipulator mechanics considering the external interactions and are widely used in various applications involving slender deformable bodies called *rods*. The flexible manipulator is modelled under the formalism of Cosserat Rod theory where the link is assumed to be inextensible and unshearable but can bend and twist. It can simulate complex deformations such as combined twisting and stretching. The static equilibrium equations of Cosserat rod are nonlinear coupled ordinary differential equations (ODEs), where numerical solution is only viable.

The second part of the thesis presents a medical application of cable driven continuum surgical manipulator specially designed for minimally invasive neurosurgical procedures. The study focusses on the designing of such manipulators based on the specific requirement of a surgical task such as manoeuvrability and stiffness. The designing process and modelling was carried out by employing Cosserat rod theory. The primary characteristics of the design are simplicity, compactness, and inherent ease of sterilization. The prototype demonstrated adequate dexterity during a basic functionality test.

The third portion of the present thesis is an exploration of the behavior of highly flexible initially curved manipulators, and beam like structures. Curved beams have inherent nonlinearity due to their initial curvature, which makes their analysis difficult. A new discretization approach is presented using piecewise clothoid elements. The clothoid curves interpolates the initial shape of the centreline of the beams in terms of piecewise linear curvatures. The use of clothoid curves for discretization is found to yield accurate results with lower computational effort. In addition, the model is simple, intuitive, and easy to implement. Incorporating the clothoid curves in the Cosserat rod model resulted in an isogeometric method, where a unified set of parameters govern the geometric design as well as static simulation.

The final portion of the thesis presents a preliminary attempt to model the dynamic behaviour of planar curved beams by incorporating piecewise clothoid interpolation into the dynamic Cosserat rod model Cosserat rod dynamic equations are a system of coupled partial

differential equations (PDEs) and are very difficult to solve. An implicit numerical solution algorithm discretizes the time variable, and a shooting method solves the resultant boundary value problem (BVP) in space variable. The effect of number of control points and the numerical damping on the dynamic response of cantilever beam systems was discussed.

Keywords: Highly flexible manipulators, continuum manipulators, Cosserat rod theory, clothoid interpolation.

सारांश

उच्च लचीले मैनिपुलेटर्स विभिन्न अनुप्रयोगों में उनकी चपलता, हल्के डिज़ाइन, और जटिल वातावरण में नेविगेट करने की क्षमता के कारण अद्वितीय लाभ प्रदान करते हैं। हालाँकि, उनकी अंतर्निहित लचीलापन मॉडलिंग, नियंत्रण, और सटीक स्थिति के मामले में महत्वपूर्ण चुनौतियाँ पेश करता है। यह थीसिस इन चुनौतियों का समाधान करने और विविध क्षेत्रों में उनके प्रभावी उपयोग को सक्षम करने के लिए उच्च लचीले मैनिपुलेटर्स की यांत्रिकी की जांच करता है। उच्च लचीले मैनिपुलेटर्स के यांत्रिक व्यवहार को समझना उनके सफल डिज़ाइन, नियंत्रण, और संचालन के लिए महत्वपूर्ण है। उनके महत्वपूर्ण विक्षेपण और जटिल गतिशील विशेषताओं के कारण, पारंपरिक कठोर शरीर मैनिपुलेटर मॉडल उनके व्यवहार को सटीक रूप से नहीं पकड़ पाते हैं। यह थीसिस उच्च लचीले मैनिपुलेटर्स की यांत्रिकी की जांच करता है, उनके गति की भविष्यवाणी करने और उन्हें इष्टतम प्रदर्शन के लिए डिज़ाइन करने के लिए मॉडल और विश्लेषण तकनीकों का विकास करता है।

इस थीसिस का पहला भाग अत्यधिक विकार्य संरचनाओं में मॉडलिंग दृष्टिकोणों का वर्णन करता है। सामान्यत:, पारंपरिक दृष्टिकोण जैसे कि यूलर-बर्नौली बीम मॉडल छोटे घुमाव की अंतर्निहित धारणाओं के कारण अत्यधिक विकृत संरचनाओं को मॉडल करने के लिए अपर्याप्त होते हैं। इसलिए, अधिक परिष्कृत तरीकों की आवश्यकता होती है जो इन धारणाओं को शिथिल करते हैं। ज्यामितीय रूप से सटीक (Geometrically exact) बीम सिद्धांत इस श्रेणी में आते हैं। मुख्यतः, निरंतर और विविक्त विधियों का उपयोग अत्यधिक विकृत निकायों को मॉडल करने के लिए किया जाता है। निरंतर विधियाँ विकृत निकायों को निरंतरता के रूप में मानती हैं और गतिशील व्यवहार का प्रतिनिधित्व करने वाले शासी आंशिक भिन्नात्मक समीकरणों को व्युत्पन्न करती हैं। इनमें से एक विधि कोस्सेराट रॉड (Cosserat rod) मॉडलिंग है, जो निरंतर मैनिपुलेटर्स को मॉडल करने में बहुत लोकप्रिय है। दूसरी ओर, विविक्त विधियाँ जैसे कि सीमित तत्व विधि (Finite element method) निरंतर संरचनाओं को एक सीमित संख्या में तत्वों में विभाजित करके मैनिपुलेटर की यांत्रिक प्रतिक्रिया के लिए एक संख्यात्मक समाधान प्राप्त करती हैं। कॉसेराट रॉड मॉडल एक मजबूत ढांचा प्रदान करता है जिससे लचीले मैनिपुलेटर की यांत्रिकी को मॉडल किया जा सकता है, बाह्य प्रभावांतरणों को ध्यान में रखते हुए, और इन्हें रॉड्स नामक पतले विकृत शरीरों को शामिल करने वाले विभिन्न अनुप्रयोगों में व्यापक रूप से उपयोग किया जाता लचीले मैनिपुलेटर को कोस्सेराट रॉड थ्योरी के आवाज में मॉडल किया गया है, जहां लिंक को अनुप्रस्थीत और अपालंबी माना गया है, लेकिन यह झुक सकता है और घुम सकता है। यह विकृतियों का अनुकरण कर सकता है जैसे कि जुड़ी हुई घुमाव और खींचाव (combined twisting and

stretching) I"कोस्सेराट रॉड के स्थिर संतुलन समीकरण गैर-रेखीय युग्मित साधारण भिन्नात्मक समीकरण (ODEs) हैं, जहां संख्यात्मक समाधान ही व्यवहार्य है।

शोध प्रबंध का दूसरा भाग न्यूनतम आक्रामक न्यूरोसर्जिकल प्रक्रियाओं के लिए विशेष रूप से डिज़ाइन किए गए केबल संचालित निरंतर सर्जिकल मैनिपुलेटर का एक चिकित्सा अनुप्रयोग प्रस्तुत करता है। अध्ययन सर्जिकल कार्य की विशिष्ट आवश्यकताओं जैसे कि संचालनशीलता और कठोरता के आधार पर ऐसे मैनिपुलेटर्स को डिज़ाइन करने पर केंद्रित है। डि-ज़ाइनिंग प्रक्रिया और मॉडलिंग को कोस्सेराट रॉड थ्योरी का उपयोग करके किया गया था। डिज़ाइन की प्राथमिक विशेषताएँ सादगी, कॉम्पैक्टनेस, और नसबंदी की अंतर्निहित सरलता हैं। प्रोटोटाइप ने एक बुनियादी कार्यक्षमता परीक्षण के दौरान पर्याप्त चपलता का प्रदर्शन किया।

वर्तमान शोध प्रबंध का अंतिम भाग अत्यधिक लचीले प्रारंभिक वक्र मैनिपुलेटर्स और बीम जैसी संरचनाओं के व्यवहार की खोज है। वक्र बीमों में प्रारंभिक वक्रता के कारण अंतर्निहित गैर-रेखीयता होती है, जिससे उनका विश्लेषण कठिन हो जाता है। एक नया विविक्तीकरण दृष्टिकोण पीसवाइज़ क्लोथोइड तत्वों का उपयोग करके प्रस्तुत किया गया है। क्लोथोइड वक्र बीमों की केंद्र रेखा के प्रारंभिक आकार को पीसवाइज़ रैखिक वक्रताओं के संदर्भ में इंटरपोलेट करता है। विविक्ती-करण के लिए क्लोथोइड वक्रों का उपयोग कम कंप्यूटेशनल प्रयास के साथ सटीक परिणाम देने के लिए पाया गया। इसके अतिरिक्त, हमारा मॉडल सरल, सहज, और लागू करने में आसान है। कोस्सेराट रॉड मॉडल में क्लोथोइड वक्रों को शामिल करना एक सममित विधि में परिणत हुआ, जहां एकीकृत मापदंडों का सेट ज्यामितीय डिज़ाइन और स्थिर अनुकरण दोनों को नियंत्रित करता है।

शोध प्रबंध को प्रारंभिक प्रयास के साथ समापन किया गया है, जो प्लानर वक्र बीमों के गतिशील व्यवहार को मॉडल करने के लिए पीसवाइज़ क्लोथोइड इंटरपोलेशन को गतिशील कोस्सेराट रॉड मॉडल में शामिल करता है। कोस्सेराट रॉड के गतिशील समीकरण युग्मित आंशिक भिन्नात्मक समीकरणों (PDEs) का एक सिस्टम होते हैं और इन्हें हल करना बहुत कठिन होता है। एक अंतर्निहित संख्यात्मक समाधान एल्गोरिदम समय चर को विविक्त करता है, और एक शूटिंग विधि अंतराल चर में परिणामी सीमा मान समस्या (BVP) को हल करती है।

मुख्य शब्द: अत्यधिक लचीले मैनिपुलेटर्स, निरंतर मैनिपुलेटर्स, कोस्सेराट रॉड थ्योरी, क्लोथोइड इंटरपोलेशन।

Contents

List of Figures

List of Tables

ABBREVIATIONS

IITD	Indian Institute of Technology Delhi
AIIMS-D	All India Institute of Medical Science Delhi
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
BVP	Boundary Value Problem
IVP	Initial Value Problem
MIS	Minimally Invasive Surgery
CAD	Computer Aided Design
FE	Finite Element
FEA	Finite Element Analysis
LFEA	Linear Finite Element Analysis
NLFEA	Nonlinear Finite Element Analysis
NURBS	Non-uniform Rational B-spline
IGA	Isogeometric Analysis
FDM	Finite Difference Method
AMM	Assumed Mode Method
LPM	Lumped Parameter Method
MSM	Mass Spring Model
MBD	Multibody Dynamics
NOC	Natural Orthogonal Complement matrix
DeNOC	Decoupled Natural Orthogonal Complement matrix
FSM	Finite Segment Method

ANCF	Absolute Nodal Coordinate Formulation
NLBT	Nonlinear Beam Theory
PCC	Piecewise Constant Curvature
PCS	Piecewise Constant Strain
MEMS	Micro Electro-Mechanical Systems
UVMS	Underwater Vehicle Manipulator System
RMSE	Root Mean Squared Error
ABS	Acrylonitrile Butadiene Styrene
CFL	Courant-Friedrichs-Lewy
BDF	Backward Differentiation Formula
2D	Two dimensional
3D	Three dimensional

NOTATION

The following rules are followed throughout this thesis for the notations:

- A **bold lower-case letter** represents a vector.
- A **bold upper-case letter** represents a tensor or a matrix.
- An *Italics letter* represents a scalar.
- Points are represented using upper-case letters.

Independent Variables

Quantity	**Description**	**Unit**
s	Arc-length parameter	(m)
t	Time	(s)

Coordinate Frames

Symbol	**Description**
O	Origin
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	Global frame
$\{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$	Director frame

Groups and Algebras

Symbol	**Description**
$\mathbb{E}^3$	Right-handed orthonormal basis
$\mathbb{R}^3$	Three-dimensional Euclidean space
SO(3)	Group of all rotations about the origin of $\mathbb{R}^3$
$\mathfrak{so}(3)$	Lie Algebra of the Lie Group SO(3)

Accents and Notation

Symbol	**Description**
$(\cdot)'$	Derivative with respect to arc-length parameter s
$\dot{(\cdot)}$	Derivative with respect to time t
$\tilde{(\cdot)}$	Mapping from $\mathbb{R}^3$ to $\mathfrak{so}(3)$ E.g., For any vector $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$, $\tilde{\mathbf{p}} \equiv \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$
$\check{(\cdot)}$	Distributed terms such as force and moment in Cosserat rod theory
$\hat{(\cdot)}$	Unit vector
$(\times)$	Cross product
$\|\|(\cdot)\|\|$	Modulus of a vector or quaternion
$\mathbf{A}^{\mathrm{T}}$	Transpose of a matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	Inverse of a matrix $\mathbf{A}$
$\mathrm{e}^{\mathbf{A}}$	Exponential of a matrix $\mathbf{A}$

Scalar quantities

Quantity	Description	Unit
L	Undeformed length of rod	m
Δs	Space step or increment in the arc-length parameter	m
ρ	Density of material of rod	kg/m^3
A	Area of cross-section of rod	m^2
E	Young's modulus of material of rod	Pa
μ	Poisson's ratio of material of rod	-
G	Shear modulus of material of rod	Pa
θ	Angle of rotation in quaternion representation	rad
q_1, q_2, q_3, q_4	Components of unit quaternion	-
I_{11}, I_{22}, I_{33}	Second moment of area along principal axes	m^4
N	Number of nodes/control points in FEA or IGA	-
N_{c}	Number of control points in clothoid interpolation	-
M	Bending moment at the tip of cable driven manipulator	Nm
T	Tension in the cable in manipulator	N
R_{offset}	Cable offset from the centreline of manipulator	m
D	Diameter of circular cross-section of rod	m
B	Breadth of rectangular cross-section of rod	m
H	Height of rectangular cross-section of rod	m
σ_b	Maximum bending stress in the cross-section of rod	Pa
y_{d}	Distance from the neutral axis to the extreme fiber	m
γ	Angle of tip-concentrated follower load	rad
R_{c}	Radius of curvature of a circular arc rod	m
c	An arbitrary constant	-
λ	A parameter used to define curves	-
α_{i}, α_{o}, β_{i}, β_{o}, μ_{i}, μ_{o}	Parameters in sandwiched beams	-
α	Parameter in BDF-α method	-
l_1, l_2	Lenght parameters of surgical manipulator	m

Vector quantities

Quantity	Description	Unit
$\mathbf{r}(s,t)$	Centreline position of rod, $\mathbf{r} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^{\mathrm{T}}$	m
$\mathbf{q}(s,t)$	Unit quaternion, $\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^{\mathrm{T}}$	-
$\hat{\mathbf{a}}$	Unit vector along axis of quaternion representation	-
$\mathbf{f}(s,t)$	Internal contact force, $\mathbf{f} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}^{\mathrm{T}}$	N
$\mathbf{n}(s,t)$	Internal contact moment, $\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^{\mathrm{T}}$	Nm
$\check{\mathbf{f}}(s,t)$	Distributed force per unit length, $\check{\mathbf{f}} = \begin{bmatrix} \check{f}_1 & \check{f}_2 & \check{f}_3 \end{bmatrix}^{\mathrm{T}}$	N/m
$\check{\mathbf{n}}(s,t)$	Distributed moment per unit length, $\check{\mathbf{n}} = \begin{bmatrix} \check{n}_1 & \check{n}_2 & \check{n}_3 \end{bmatrix}^{\mathrm{T}}$	Nm/m
$\mathbf{v}(s,t)$	Strain vector, $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^{\mathrm{T}}$	-
$\mathbf{k}(s,t)$	Curvature vector in the local frame, $\mathbf{k} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^{\mathrm{T}}$	1/m
$\mathbf{u}(s,t)$	Velocity in the local frame, $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{\mathrm{T}}$	m/s
$\mathbf{w}(s,t)$	Angular velocity in the local frame, $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^{\mathrm{T}}$	1/s
$\mathbf{v}^*(s,t)$	Reference strain, $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$ for a straight rod	-
$\mathbf{k}^*(s,t)$	Reference curvature $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ for a straight rod	1/m
$\mathbf{y}$	Initial guess for solution of rod equations	-
$\delta\mathbf{y}$	Increment in solution of rod equations	-
$\mathbf{z}$	Residual vector	-
$\mathbf{h}$	A function which evaluates residual	-
$\hat{\mathbf{k}}$	A unit vector in the direction of $\mathbf{k}$	-
$\mathbf{0}$	A 3×1 null vector	-

Matrices

Quantity	Description	Unit
$\mathbf{R}(s,t)$	Rotation matrix for material orientation	-
$\mathbf{B}$	A 4×3 matrix for quaternion derivative $$\mathbf{B} \equiv 0.5 \begin{bmatrix} -q_2 & -q_3 & -q_4 \\ q_1 & -q_4 & q_3 \\ q_4 & q_1 & -q_2 \\ -q_3 & q_2 & q_1 \end{bmatrix}$$	-
$\mathbf{J}$	Second mass moment of inertia tensor $$\mathbf{J} \equiv \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$$	m^4
$\mathbf{K}_{\mathrm{se}}$	Stiffness matrix for shear and extension $$\mathbf{K}_{\mathrm{se}} \equiv \begin{bmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{bmatrix}$$	N
$\mathbf{K}_{\mathrm{bt}}$	Stiffness matrix for bending and torsion $$\mathbf{K}_{\mathrm{bt}} \equiv \begin{bmatrix} EI_{11} & 0 & 0 \\ 0 & EI_{22} & 0 \\ 0 & 0 & GI_{33} \end{bmatrix}$$	Nm^2
$\mathbf{B}_{\mathrm{se}}$	Damping matrix for shear and extension	Ns
$\mathbf{B}_{\mathrm{bt}}$	Damping matrix for bending and twisting	$\mathrm{Nm}^2\mathrm{s}$
$\mathbf{J}_s$	Jacobian matrix	-
$\mathbf{I}_{3\times 3}$	A 3×3 identity matrix	-