

# **MECHANICS OF HIGHLY FLEXIBLE MANIPULATORS**

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**DEPARTMENT OF MECHANICAL ENGINEERING  
INDIAN INSTITUTE OF TECHNOLOGY DELHI  
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# **MECHANICS OF HIGHLY FLEXIBLE MANIPULATORS**

by

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**DEPARTMENT OF MECHANICAL ENGINEERING**

**submitted**

**in fulfilment of the requirements of the degree of doctor of philosophy**

**to the**



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# CERTIFICATE

This is to certify that the thesis titled **Mechanics of Highly Flexible Manipulators**, submitted by **Sreejath. S**, to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy**, is a bona fide record of the research work done by him under our supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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# ABSTRACT

Highly flexible manipulators offer unique advantages in various applications due to their dexterity, lightweight design, and ability to navigate complex environments. However, their inherent flexibility presents significant challenges in terms of modeling, control, and accurate positioning. This thesis investigates the mechanics of highly flexible manipulators to address these challenges and enable their effective utilization in diverse fields. Understanding the mechanical behavior of highly flexible manipulators is crucial for their successful design, control, and operation. Due to their significant deflection and complex dynamic characteristics, traditional rigid-body manipulator models fail to capture their behavior accurately. This thesis explores the mechanics of highly flexible manipulators, developing models and analysis techniques to predict their motion and design them for optimal performance.

First portion of the thesis describes the modelling approaches in highly deformable structures. Generally, conventional approaches such as Euler-Bernoulli beam models are insufficient to model highly deformable structures due to their inherent assumptions of small rotation. Hence more sophisticated methods are required which relaxes the assumptions. Geometrically-exact beam theories are of such category. Mainly, continuous, and discrete methods are employed to model highly deformable bodies. Continuous methods treat deformable bodies as continuum and derive the governing partial differential equations that represent the dynamic behaviours. One of such methods is Cosserat rod modelling, which is very popular in modelling continuum manipulators. On the other hand, discrete methods such as the Finite Element Method (FEM) discretize the continuum structures into a finite number of elements to obtain a numerical solution for the manipulator's mechani-

cal response. Cosserat rod model offers a robust framework to model flexible manipulator mechanics considering the external interactions and are widely used in various applications involving slender deformable bodies called *rods*. The flexible manipulator is modelled under the formalism of Cosserat Rod theory where the link is assumed to be inextensible and un-shearable but can bend and twist. It can simulate complex deformations such as combined twisting and stretching. The static equilibrium equations of Cosserat rod are nonlinear coupled ordinary differential equations (ODEs), where numerical solution is only viable.

The second part of the thesis presents a medical application of cable driven continuum surgical manipulator specially designed for minimally invasive neurosurgical procedures. The study focusses on the designing of such manipulators based on the specific requirement of a surgical task such as manoeuvrability and stiffness. The designing process and modelling was carried out by employing Cosserat rod theory. The primary characteristics of the design are simplicity, compactness, and inherent ease of sterilization. The prototype demonstrated adequate dexterity during a basic functionality test.

The third portion of the present thesis is an exploration of the behavior of highly flexible initially curved manipulators, and beam like structures. Curved beams have inherent nonlinearity due to their initial curvature, which makes their analysis difficult. A new discretization approach is presented using piecewise clothoid elements. The clothoid curves interpolates the initial shape of the centreline of the beams in terms of piecewise linear curvatures. The use of clothoid curves for discretization is found to yield accurate results with lower computational effort. In addition, the model is simple, intuitive, and easy to implement. Incorporating the clothoid curves in the Cosserat rod model resulted in an isogeometric method, where a unified set of parameters govern the geometric design as well as static simulation.

The final portion of the thesis presents a preliminary attempt to model the dynamic behaviour of planar curved beams by incorporating piecewise clothoid interpolation into the dynamic Cosserat rod model. Cosserat rod dynamic equations are a system of coupled partial

differential equations (PDEs) and are very difficult to solve. An implicit numerical solution algorithm discretizes the time variable, and a shooting method solves the resultant boundary value problem (BVP) in space variable. The effect of number of control points and the numerical damping on the dynamic response of cantilever beam systems was discussed.

**Keywords:** Highly flexible manipulators, continuum manipulators, Cosserat rod theory, clothoid interpolation.

# सारांश

उच्च लचीले मैनिपुलेटर्स विभिन्न अनुप्रयोगों में उनकी चपलता, हल्के डिज़ाइन, और जटिल वातावरण में नेविगेट करने की क्षमता के कारण अद्वितीय लाभ प्रदान करते हैं। हालाँकि, उनकी अंतर्निहित लचीलापन मॉडलिंग, नियंत्रण, और सटीक स्थिति के मामले में महत्वपूर्ण चुनौतियाँ पेश करता है। यह थीसिस इन चुनौतियों का समाधान करने और विविध क्षेत्रों में उनके प्रभावी उपयोग को सक्षम करने के लिए उच्च लचीले मैनिपुलेटर्स की यांत्रिकी की जांच करता है। उच्च लचीले मैनिपुलेटर्स के यांत्रिक व्यवहार को समझना उनके सफल डिज़ाइन, नियंत्रण, और संचालन के लिए महत्वपूर्ण है। उनके महत्वपूर्ण विक्षेपण और जटिल गतिशील विशेषताओं के कारण, पारंपरिक कठोर शरीर मैनिपुलेटर मॉडल उनके व्यवहार को सटीक रूप से नहीं पकड़ पाते हैं। यह थीसिस उच्च लचीले मैनिपुलेटर्स की यांत्रिकी की जांच करता है, उनके गति की भविष्यवाणी करने और उन्हें इष्टतम प्रदर्शन के लिए डिज़ाइन करने के लिए मॉडल और विश्लेषण तकनीकों का विकास करता है।

इस थीसिस का पहला भाग अत्यधिक विकार्य संरचनाओं में मॉडलिंग दृष्टिकोणों का वर्णन करता है। सामान्यतः, पारंपरिक दृष्टिकोण जैसे कि यूलर-बर्नौली बीम मॉडल छोटे घुमाव की अंतर्निहित धारणाओं के कारण अत्यधिक विकृत संरचनाओं को मॉडल करने के लिए अपर्याप्त होते हैं। इसलिए, अधिक परिष्कृत तरीकों की आवश्यकता होती है जो इन धारणाओं को शिथिल करते हैं। ज्यामितीय रूप से सटीक (Geometrically exact) बीम सिद्धांत इस श्रेणी में आते हैं। मुख्यतः, निरंतर और विविक्त विधियों का उपयोग अत्यधिक विकृत निकायों को मॉडल करने के लिए किया जाता है। निरंतर विधियाँ विकृत निकायों को निरंतरता के रूप में मानती हैं और गतिशील व्यवहार का प्रतिनिधित्व करने वाले शासी आंशिक भिन्नात्मक समीकरणों को व्युत्पन्न करती हैं। इनमें से एक विधि कोस्सेराट रॉड (Cosserat rod) मॉडलिंग है, जो निरंतर मैनिपुलेटर्स को मॉडल करने में बहुत लोकप्रिय है। दूसरी ओर, विविक्त विधियाँ जैसे कि सीमित तत्व विधि (Finite element method) निरंतर संरचनाओं को एक सीमित संख्या में तत्वों में विभाजित करके मैनिपुलेटर की यांत्रिक प्रतिक्रिया के लिए एक संख्यात्मक समाधान प्राप्त करती हैं। कॉसेराट रॉड मॉडल एक मजबूत ढांचा प्रदान करता है जिससे लचीले मैनिपुलेटर की यांत्रिकी को मॉडल किया जा सकता है, बाह्य प्रभावांतरणों को ध्यान में रखते हुए, और इन्हें रॉड्स नामक पतले विकृत शरीरों को शामिल करने वाले विभिन्न अनुप्रयोगों में व्यापक रूप से उपयोग किया जाता लचीले मैनिपुलेटर को कोस्सेराट रॉड थ्योरी के आवाज में मॉडल किया गया है, जहां लिंक को अनुप्रस्थीत और अपालंबी माना गया है, लेकिन यह झुक सकता है और घुम सकता है। यह विकृतियों का अनुकरण कर सकता है जैसे कि जुड़ी हुई घुमाव और खींचाव (combined twisting and

stretching)। "कोस्सेराट रॉड के स्थिर संतुलन समीकरण गैर-रेखीय युग्मित साधारण भिन्नात्मक समीकरण (ODEs) हैं, जहां संख्यात्मक समाधान ही व्यवहार्य है।

शोध प्रबंध का दूसरा भाग न्यूनतम आक्रामक न्यूरोसर्जिकल प्रक्रियाओं के लिए विशेष रूप से डिज़ाइन किए गए केबल संचालित निरंतर सर्जिकल मैनिपुलेटर का एक चिकित्सा अनुप्रयोग प्रस्तुत करता है। अध्ययन सर्जिकल कार्य की विशिष्ट आवश्यकताओं जैसे कि संचालनशीलता और कठोरता के आधार पर ऐसे मैनिपुलेटर्स को डिज़ाइन करने पर केंद्रित है। डिज़ाइनिंग प्रक्रिया और मॉडलिंग को कोस्सेराट रॉड थ्योरी का उपयोग करके किया गया था। डिज़ाइन की प्राथमिक विशेषताएँ सादगी, कॉम्पैक्टनेस, और नसबंदी की अंतर्निहित सरलता हैं। प्रोटोटाइप ने एक बुनियादी कार्यक्षमता परीक्षण के दौरान पर्याप्त चपलता का प्रदर्शन किया।

वर्तमान शोध प्रबंध का अंतिम भाग अत्यधिक लचीले प्रारंभिक वक्र मैनिपुलेटर्स और बीम जैसी संरचनाओं के व्यवहार की खोज है। वक्र बीमों में प्रारंभिक वक्रता के कारण अंतर्निहित गैर-रेखीयता होती है, जिससे उनका विश्लेषण कठिन हो जाता है। एक नया विविक्तीकरण दृष्टिकोण पीसवाइज़ क्लोथोइड तत्वों का उपयोग करके प्रस्तुत किया गया है। क्लोथोइड वक्र बीमों की केंद्र रेखा के प्रारंभिक आकार को पीसवाइज़ रैखिक वक्रताओं के संदर्भ में इंटरपोलेट करता है। विविक्तीकरण के लिए क्लोथोइड वक्रों का उपयोग कम कंप्यूटेशनल प्रयास के साथ सटीक परिणाम देने के लिए पाया गया। इसके अतिरिक्त, हमारा मॉडल सरल, सहज, और लागू करने में आसान है। कोस्सेराट रॉड मॉडल में क्लोथोइड वक्रों को शामिल करना एक सममित विधि में परिणत हुआ, जहां एकीकृत मापदंडों का सेट ज्यामितीय डिज़ाइन और स्थिर अनुकरण दोनों को नियंत्रित करता है।

शोध प्रबंध को प्रारंभिक प्रयास के साथ समापन किया गया है, जो प्लानर वक्र बीमों के गतिशील व्यवहार को मॉडल करने के लिए पीसवाइज़ क्लोथोइड इंटरपोलेशन को गतिशील कोस्सेराट रॉड मॉडल में शामिल करता है। कोस्सेराट रॉड के गतिशील समीकरण युग्मित आंशिक भिन्नात्मक समीकरणों (PDEs) का एक सिस्टम होते हैं और इन्हें हल करना बहुत कठिन होता है। एक अंतर्निहित संख्यात्मक समाधान एल्गोरिदम समय चर को विविक्त करता है, और एक शूटिंग विधि अंतराल चर में परिणामी सीमा मान समस्या (BVP) को हल करती है।

**मुख्य शब्द:** अत्यधिक लचीले मैनिपुलेटर्स, निरंतर मैनिपुलेटर्स, कोस्सेराट रॉड थ्योरी, क्लोथोइड इंटरपोलेशन।

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# ABBREVIATIONS

<b>IITD</b>	Indian Institute of Technology Delhi
<b>AIIMS-D</b>	All India Institute of Medical Science Delhi
<b>ODE</b>	Ordinary Differential Equation
<b>PDE</b>	Partial Differential Equation
<b>BVP</b>	Boundary Value Problem
<b>IVP</b>	Initial Value Problem
<b>MIS</b>	Minimally Invasive Surgery
<b>CAD</b>	Computer Aided Design
<b>FE</b>	Finite Element
<b>FEA</b>	Finite Element Analysis
<b>LFEA</b>	Linear Finite Element Analysis
<b>NLFEA</b>	Nonlinear Finite Element Analysis
<b>NURBS</b>	Non-uniform Rational B-spline
<b>IGA</b>	Isogeometric Analysis
<b>FDM</b>	Finite Difference Method
<b>AMM</b>	Assumed Mode Method
<b>LPM</b>	Lumped Parameter Method
<b>MSM</b>	Mass Spring Model
<b>MBD</b>	Multibody Dynamics
<b>NOC</b>	Natural Orthogonal Complement matrix
<b>DeNOC</b>	Decoupled Natural Orthogonal Complement matrix
<b>FSM</b>	Finite Segment Method

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<b>ANCF</b>	Absolute Nodal Coordinate Formulation
<b>NLBT</b>	Nonlinear Beam Theory
<b>PCC</b>	Piecewise Constant Curvature
<b>PCS</b>	Piecewise Constant Strain
<b>MEMS</b>	Micro Electro-Mechanical Systems
<b>UVMS</b>	Underwater Vehicle Manipulator System
<b>RMSE</b>	Root Mean Squared Error
<b>ABS</b>	Acrylonitrile Butadiene Styrene
<b>CFL</b>	Courant-Friedrichs-Lewy
<b>BDF</b>	Backward Differentiation Formula
<b>2D</b>	Two dimensional
<b>3D</b>	Three dimensional

# NOTATION

The following rules are followed throughout this thesis for the notations:

- A **bold lower-case letter** represents a vector.
- A **bold upper-case letter** represents a tensor or a matrix.
- An *Italics letter* represents a scalar.
- Points are represented using upper-case letters.

## Independent Variables

Quantity	Description	Unit
$s$	Arc-length parameter	(m)
$t$	Time	(s)

## Coordinate Frames

Symbol	Description
O	Origin
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	Global frame
$\{\mathbf{d}_1(s), \mathbf{d}_2(s), \mathbf{d}_3(s)\}$	Director frame

## Groups and Algebras

Symbol	Description
$\mathbb{E}^3$	Right-handed orthonormal basis
$\mathbb{R}^3$	Three-dimensional Euclidean space
$\text{SO}(3)$	Group of all rotations about the origin of $\mathbb{R}^3$
$\mathfrak{so}(3)$	Lie Algebra of the Lie Group $\text{SO}(3)$

## Accents and Notation

Symbol	Description
$(\cdot)'$	Derivative with respect to arc-length parameter $s$
$\dot{(\cdot)}$	Derivative with respect to time $t$
$\tilde{(\cdot)}$	Mapping from $\mathbb{R}^3$ to $\mathfrak{so}(3)$ E.g., For any vector $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ , $\tilde{\mathbf{p}} \equiv \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$
$\ddot{(\cdot)}$	Distributed terms such as force and moment in Cosserat rod theory
$\hat{(\cdot)}$	Unit vector
$(\times)$	Cross product
$\ (\cdot)\ $	Modulus of a vector or quaternion
$\mathbf{A}^T$	Transpose of a matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	Inverse of a matrix $\mathbf{A}$
$e^{\mathbf{A}}$	Exponential of a matrix $\mathbf{A}$

## Scalar quantities

Quantity	Description	Unit
$L$	Undeformed length of rod	m
$\Delta s$	Space step or increment in the arc-length parameter	m
$\rho$	Density of material of rod	kg/m <sup>3</sup>
$A$	Area of cross-section of rod	m <sup>2</sup>
$E$	Young's modulus of material of rod	Pa
$\mu$	Poisson's ratio of material of rod	-
$G$	Shear modulus of material of rod	Pa
$\theta$	Angle of rotation in quaternion representation	rad
$q_1, q_2, q_3, q_4$	Components of unit quaternion	-
$I_{11}, I_{22}, I_{33}$	Second moment of area along principal axes	m <sup>4</sup>
$N$	Number of nodes/control points in FEA or IGA	-
$N_c$	Number of control points in clothoid interpolation	-
$M$	Bending moment at the tip of cable driven manipulator	Nm
$T$	Tension in the cable in manipulator	N
$R_{\text{offset}}$	Cable offset from the centreline of manipulator	m
$D$	Diameter of circular cross-section of rod	m
$B$	Breadth of rectangular cross-section of rod	m
$H$	Height of rectangular cross-section of rod	m
$\sigma_b$	Maximum bending stress in the cross-section of rod	Pa
$y_d$	Distance from the neutral axis to the extreme fiber	m
$\gamma$	Angle of tip-concentrated follower load	rad
$R_c$	Radius of curvature of a circular arc rod	m
$c$	An arbitrary constant	-
$\lambda$	A parameter used to define curves	-
$\alpha_i, \alpha_o, \beta_i, \beta_o, \mu_i, \mu_o$	Parameters in sandwiched beams	-
$\alpha$	Parameter in BDF- $\alpha$ method	-
$l_1, l_2$	Length parameters of surgical manipulator	m

## Vector quantities

Quantity	Description	Unit
$\mathbf{r}(s, t)$	Centreline position of rod, $\mathbf{r} = [r_1 \ r_2 \ r_3]^T$	m
$\mathbf{q}(s, t)$	Unit quaternion, $\mathbf{q} = [q_1 \ q_2 \ q_3 \ q_4]^T$	-
$\hat{\mathbf{a}}$	Unit vector along axis of quaternion representation	-
$\mathbf{f}(s, t)$	Internal contact force, $\mathbf{f} = [f_1 \ f_2 \ f_3]^T$	N
$\mathbf{n}(s, t)$	Internal contact moment, $\mathbf{n} = [n_1 \ n_2 \ n_3]^T$	Nm
$\check{\mathbf{f}}(s, t)$	Distributed force per unit length, $\check{\mathbf{f}} = [\check{f}_1 \ \check{f}_2 \ \check{f}_3]^T$	N/m
$\check{\mathbf{n}}(s, t)$	Distributed moment per unit length, $\check{\mathbf{n}} = [\check{n}_1 \ \check{n}_2 \ \check{n}_3]^T$	Nm/m
$\mathbf{v}(s, t)$	Strain vector, $\mathbf{v} = [v_1 \ v_2 \ v_3]^T$	-
$\mathbf{k}(s, t)$	Curvature vector in the local frame, $\mathbf{k} = [k_1 \ k_2 \ k_3]^T$	1/m
$\mathbf{u}(s, t)$	Velocity in the local frame, $\mathbf{u} = [u_1 \ u_2 \ u_3]^T$	m/s
$\mathbf{w}(s, t)$	Angular velocity in the local frame, $\mathbf{w} = [w_1 \ w_2 \ w_3]^T$	1/s
$\mathbf{v}^*(s, t)$	Reference strain, $[0 \ 0 \ 1]^T$ for a straight rod	-
$\mathbf{k}^*(s, t)$	Reference curvature $[0 \ 0 \ 0]^T$ for a straight rod	1/m
$\mathbf{y}$	Initial guess for solution of rod equations	-
$\delta\mathbf{y}$	Increment in solution of rod equations	-
$\mathbf{z}$	Residual vector	-
$\mathbf{h}$	A function which evaluates residual	-
$\hat{\mathbf{k}}$	A unit vector in the direction of $\mathbf{k}$	-
$\mathbf{0}$	A $3 \times 1$ null vector	-

## Matrices

Quantity	Description	Unit
$\mathbf{R}(s, t)$	Rotation matrix for material orientation	-
$\mathbf{B}$	A $4 \times 3$ matrix for quaternion derivative	
	$\mathbf{B} \equiv 0.5 \begin{bmatrix} -q_2 & -q_3 & -q_4 \\ q_1 & -q_4 & q_3 \\ q_4 & q_1 & -q_2 \\ -q_3 & q_2 & q_1 \end{bmatrix}$	-
$\mathbf{J}$	Second mass moment of inertia tensor	
	$\mathbf{J} \equiv \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix}$	$\text{m}^4$
$\mathbf{K}_{\text{se}}$	Stiffness matrix for shear and extension	
	$\mathbf{K}_{\text{se}} \equiv \begin{bmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{bmatrix}$	N
$\mathbf{K}_{\text{bt}}$	Stiffness matrix for bending and torsion	
	$\mathbf{K}_{\text{bt}} \equiv \begin{bmatrix} EI_{11} & 0 & 0 \\ 0 & EI_{22} & 0 \\ 0 & 0 & GI_{33} \end{bmatrix}$	$\text{Nm}^2$
$\mathbf{B}_{\text{se}}$	Damping matrix for shear and extension	Ns
$\mathbf{B}_{\text{bt}}$	Damping matrix for bending and twisting	$\text{Nm}^2\text{s}$
$\mathbf{J}_s$	Jacobian matrix	-
$\mathbf{I}_{3 \times 3}$	A $3 \times 3$ identity matrix	-