

**FINITE ELEMENT METHODS FOR FRICTIONAL  
AND FRICTIONLESS CONTACT PROBLEMS**

**TANVI WADHAWAN**



**DEPARTMENT OF MATHEMATICS  
INDIAN INSTITUTE OF TECHNOLOGY DELHI  
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AND FRICTIONLESS CONTACT PROBLEMS**

by

**TANVI WADHAWAN**

**DEPARTMENT OF MATHEMATICS**

submitted

in fulfilment of the requirements of the degree of Doctor of Philosophy

to the



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*Dedicated to  
my grandfather*



# Certificate

This is to certify that the thesis entitled **FINITE ELEMENT METHODS FOR FRICTIONAL AND FRICTIONLESS CONTACT PROBLEMS** submitted by **Ms. Tanvi Wadhawan** to the **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi  
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**Dr. Kamana Porwal**  
Research Advisor



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Tanvi Wadhawan



# Abstract

The principal aim of this thesis is to investigate finite element methods applied to both frictional and frictionless contact problems. Contact problems play a central role in a wide range of mechanical structures and hold significant importance in various fields such as hydrostatics and thermostatics. Within this study, we emphasise on the contact problems characterized by various conditions, including the Signorini non-penetration condition and normal compliance condition while the frictional force is modeled using Coulomb's law of dry friction and Tresca friction laws. The variational inequalities, which represent the variational formulations of contact problems, have provided a powerful tool in the mathematical analysis of these models. In this thesis, we examine both elliptic and parabolic variational inequalities. Within the context of elliptic variational inequalities, our investigation encompasses the Signorini contact problem and the frictional contact problem with normal compliance. In the realm of parabolic variational inequalities, we conduct a detailed analysis of the quasi-static contact problem.

The regularity of the solution plays a pivotal role in determining the qualitative behavior of the approximation. However, there are situations where the solution to the variational inequalities exhibit irregularities, stemming from factors like the presence of free boundaries, reentrant corners within the domain etc. Due to the inherent non-smooth and non-linear nature of variational inequalities, there has been a growing interest in *a posteriori* error control for these variational inequalities and discretization methods in recent years. A pivotal step in the development of adaptive finite element methods is the derivation of *a posteriori* error estimators. These estimators establish a connection between the error and variables that can be computed using the discrete solution and the available data. In the pursuit of reducing computational costs without compromising the accuracy of the numerical solution, adaptive mesh generation emerges as a highly advantageous approach. To this end, a rigorous theoretical framework is developed to establish these *a posteriori* error estimates and obtain convergence rates for

some variational inequalities. These estimates serve to quantify the accuracy of the numerical approximation and provide essential insights for evaluating the reliability of the computed solutions. The theoretical discoveries are substantiated by the numerical investigations.

This thesis comprises of seven chapters. The first chapter of the thesis lays the foundation for the subsequent chapters by conducting an extensive review and analysis of relevant literature. The next two chapters of the thesis delve into the derivation of residual-based *a posteriori* error estimates for the unilateral contact problem. Specifically, quadratic finite elements are employed in the conforming setting in the first chapter, while the second chapter focuses on the discontinuous Galerkin (DG) methods. The analysis primarily hinges on the utilization of several intermediate operators, which in turn, confirms the reliability and efficiency of the error estimators. Furthermore, in Chapter 3, we establish optimal *a priori* error estimates in DG norm with the appropriate regularity assumption on the exact solution. Subsequently, Chapter 4 is devoted to the derivation of reliable and efficient pointwise *a posteriori* estimates for the Signorini problem, utilizing quadratic DG methods. The analysis primarily relies on the existence of super and sub-solutions corresponding to the continuous solution  $\mathbf{u}$  and standard *a priori* error estimates for the Green's matrix of the divergence type operator. The numerical analysis of the frictional contact problem with normal compliance in DG setting is studied in Chapter 5. In this chapter, we have also established the medius analysis along with *a posteriori* error analysis. An enriching map that bridges the DG finite element space with the conforming finite element space has been constructed and used extensively in the analysis of Chapter 3, Chapter 4 and Chapter 5. Additionally, a common property shared by various DG methods aids in formulating the analysis within a unified framework. In Chapter 6, we consider the numerical solution of a non-linear evolutionary variational inequality, arising in the study of quasi-static contact problem. Therein we study spatially semi-discrete and fully-discrete schemes for the quasi-static contact problem using non-conforming discretization in space and finite difference discretization in time. In order to validate the theoretical results, numerical results are presented at the end of each chapter from Chapter 2 to Chapter 6. Finally in the last chapter, we conclude the thesis and propose potential directions for the future work.

## सार

इस थीसिस का मुख्य उद्देश्य फ्रिक्शनल और फ्रिक्शनलेस संपर्क समस्याओं पर लागू समाप्ति तत्व विधियों की जांच करना है। संपर्क समस्याएँ यांत्रिक संरचनाओं के एक विस्तृत श्रेणी में महत्वपूर्ण भूमिका निभाती हैं और हाइड्रोस्टेटिक्स और थर्मोस्टेटिक्स जैसे विभिन्न क्षेत्रों में महत्वपूर्ण महत्व रखती हैं। इस अध्ययन में, हम विभिन्न स्थितियों द्वारा चरित्रित संपर्क समस्याओं पर जोर देते हैं, जिनमें सिग्नोरिनी गैर-प्रवेश स्थिति और सामान्य संरचना स्थिति शामिल हैं, जबकि फ्रिक्शनल बल को कूलॉब के सुखे फ्रिक्शन और ट्रेस्का फ्रिक्शन कानून का उपयोग करके मॉडल किया जाता है। संपर्क समस्याओं के वैरिएशनल असमानताएँ, जो संपर्क समस्याओं के वैरिएशनल रूपांतरणों को प्रतिनिधित्व करती हैं, इन मॉडलों के गणितीय विश्लेषण में एक शक्तिशाली उपकरण प्रदान करती हैं। इस थीसिस में, हम दोनों एल्लिप्टिक और पैराबोलिक वैरिएशनल असमानताओं का परीक्षण करते हैं। एल्लिप्टिक वैरिएशनल असमानताओं के संदर्भ में, हमारा अध्ययन सिग्नोरिनी संपर्क समस्या और सामान्य संपर्क समस्या को समावेश करता है। पैराबोलिक वैरिएशनल असमानताओं के क्षेत्र में, हम तकनीकी रूप से क्वासी-स्थिर संपर्क समस्या के विस्तृत विश्लेषण करते हैं।

समाधान की नियमितता अनुमानित क्वालिटीटिव व्यवहार को निर्धारित करने में महत्वपूर्ण भूमिका निभाती है। हालांकि, कुछ परिस्थितियों में वैरिएशनल असमानताओं का समाधान अनियमितताओं का प्रदर्शन करता है, जो क्षेत्र में निःशुल्क सीमाओं, दोहरे कोनों आदि के कारण से हो सकता है। वैरिएशनल असमानताओं की स्वाभाविक अनंग्य और गैर-लीनियर प्रकृति के कारण, हाल ही में इन वैरिएशनल असमानताओं और डिस्क्रेटाइज़ेशन विधियों के लिए अपोस्टेरियोरी त्रुटि नियंत्रण में एक बढ़ती हुई रुचि रही है। समुचित तरीके से अनुकूलनिक तत्व विकसित करने में एक प्रमुख कदम अपोस्टेरियोरी त्रुटि अंशकों का निर्धारण है। ये अंशक त्रुटि और उपलब्ध डेटा का उपयोग करके गणितीय समाधान और उपलब्ध डेटा का उपयोग करते हैं। गणना की मेहनत कम करने के लिए, बिना

संख्यात्मक समाधान की गुणवत्ता पर कोई असर नहीं पड़ता, कंप्यूटेशनल लागत को कम करने का एक अत्यधिक लाभकारी उपाय उभरता है। इस उद्देश्य के लिए, इन अपोस्टेरियोरी त्रुटि अनुमानों को स्थापित करने और कुछ वैरिएशनल असमानताओं के लिए संगति दरें प्राप्त करने के लिए एक सख्त गणितीय सिद्धांतिक ढांचा विकसित किया गया है। ये अनुमान गणितीय अनुमान की सटीकता को मापते हैं और गणितीय समाधान के विश्वसनीयता का मूल्यांकन करने के लिए आवश्यक अद्यतन देते हैं। नांतरिक अनुसंधान द्वारा सिद्धांतिक खोज से प्रमाणित किए जाते हैं।

यह थीसिस सात अध्यायों से बनी है। थीसिस का पहला अध्याय अन्य अध्यायों के लिए मौलिकता स्थापित करता है और संबंधित साहित्य का विस्तृत समीक्षा और विश्लेषण करके आगे के अध्यायों के लिए आधार रखता है। थीसिस के अगले दो अध्याय एक प्रतिबंधी संपर्क समस्या के लिए अपोस्टेरियोरी त्रुटि अंशकों की निर्धारण में डुबकते हैं। विशेष रूप से, पहले अध्याय में समानुपातिक वातावरण में क्वाड्रेटिक अंत मानों का उपयोग किया गया है, जबकि दूसरा अध्याय असमान गैलरकिन (डीजी) विधियों पर ध्यान केंद्रित करता है। विश्लेषण मुख्य रूप से कई मध्यवर्ती ऑपरेटर्स का उपयोग पर निर्भर करता है, जो फिर, त्रुटि अंशकों की विश्वसनीयता और प्रभावकारिता को साबित करता है। इसके अलावा, अध्याय 3 में, हम उत्कृष्ट पूर्व त्रुटि अंशकों की निर्धारण में डीजी नार्म में उचितता अनुमान करते हैं, अनुमान की विशेष शर्त पर। इसके बाद, अध्याय 4 में, हम सिग्नोरिनी समस्या के लिए विश्वसनीय और कुशल अपोस्टेरियोरी अनुमानों की निर्धारण में क्वाड्रेटिक डीजी विधियों का उपयोग करते हैं। विश्लेषण मुख्य रूप से निरंतर समाधान  $u$  के लिए अस्तित्व की मौजूदगी पर निर्भर करता है और डाइवर्जेंस प्रकार के ऑपरेटर के हरी के मैट्रिक्स के लिए मानक पूर्व त्रुटि अंशकों की निर्धारण की विशेष शर्त। अध्याय 5 में नॉर्मल कंप्लायंस के साथ घर्षणात्मक संपर्क समस्या का संख्यात्मक विश्लेषण डीजी सेटिंग में अध्ययन किया गया है। इस अध्याय में, हमने भविष्यवाणी विश्लेषण के साथ समन्वय विश्लेषण भी स्थापित किया है। एक ऐसा समृद्ध मानचित्र जो डीजी अंत मान क्षेत्र को समानांतर अंत मान क्षेत्र के साथ जोड़ता है, अध्याय 3, अध्याय 4 और अध्याय 5 के विश्लेषण में व्यापक रूप से उपयोग किया गया है। साथ ही, विभिन्न डीजी विधियों द्वारा साझा की गई एक सामान्य गुणधर्म एक समूचे संरचना के भीतर विश्लेषण को रूपांतरित करने में मदद करता है। अध्याय 6 में, हम एक गैर-लीनियर वैकल्पिक वैरिएशनल असमानता का संख्यात्मक समाधान का विचार करते हैं, जो प्रासंगिक स्थिर संपर्क समस्या के अध्ययन से उत्पन्न होता है। वहाँ हम अक्षग्राही और पूरी तरह से असंगत अंत मानों के संख्यात्मक और पूरी तरह से असंगत योजनाओं का अध्ययन करते हैं, स्थान में गैर-संगत विभाजन और समय में सीमित अंत मान की विभाजन। सिद्धांतिक परिणामों की पुष्टि करने के लिए, प्रायोगिक परिणामों को अध्याय 2 से अध्याय 6 तक प्रस्तुत किया गया है। अंत में आखिरी अध्याय में, हम थीसिस का संक्षिप्तीकरण करते हैं और भविष्य के कार्य के लिए संभावित दिशानिर्देश प्रस्तुत करते हैं।

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# List of Symbols

Symbol	Meaning
$\mathcal{T}_h$	regular simplicial triangulation of domain $\Omega$ ,
$T$	an element in $\mathcal{T}_h$ ,
$ T $	volume of an element $T$ ,
$\partial\Omega$	boundary of $\Omega$ ,
$\Gamma_D$	Dirichlet boundary,
$\Gamma_N$	Neumann boundary,
$\Gamma_C$	potential contact boundary,
$e$	an edge of $\mathcal{T}_h$ ,
$h_T$	diameter of the triangle $T$ ,
$h$	$\max \{h_T : T \in \mathcal{T}_h\}$ ,
$h_e$	length of an edge $e$ ,
$p$	a node of an element $T$ ,
$\mathcal{V}_h$	set of all vertices of $\mathcal{T}_h$ ,
$\mathcal{M}_h$	set of all midpoint of edges in $\mathcal{T}_h$ ,
$\mathcal{E}_h$	set of all edges of $\mathcal{T}_h$ ,
$\mathcal{V}_e$	set of two vertices of edge $e$ ,
$\mathcal{M}_e$	midpoint of edge $e$ ,
$\mathcal{V}_h^i$	set of all vertices in $\mathcal{T}_h$ that are in $\Omega$ ,
$\mathcal{V}_h^b$	set of all vertices of $\mathcal{T}_h$ that are on $\partial\Omega$ ,
$\mathcal{M}_h^i$	set of midpoints of interior edges in $\mathcal{T}_h$ ,
$\mathcal{V}_T$	set of all vertices of the triangle $T$ ,
$\mathcal{M}_T$	set of midpoints of the triangle $T$ ,

$\mathcal{T}_h^C$	set of all triangles $T$ such that $\partial T \cap \Gamma_C$ is non-empty,
$\mathcal{E}_T^C$	set of contact edges of the triangle $T \in \mathcal{T}_h^C$ ,
$\mathcal{M}_T^C$	set of midpoint of the edges $e \in \mathcal{E}_T^C$ ,
$\mathcal{E}_h^i$	set of all interior edges of $\mathcal{T}_h$ ,
$\mathcal{E}_h^b$	set of all boundary edges of $\mathcal{T}_h$ ,
$\mathcal{E}_h^D$	set of all edges lying on $\Gamma_D$ ,
$\mathcal{E}_h^N$	set of all boundary edges lying on $\Gamma_N$ ,
$\mathcal{E}_h^C$	set of all edges lying on $\Gamma_C$ ,
$\mathcal{E}_h^0$	$\mathcal{E}_h^i \cup \mathcal{E}_h^D$ ,
$\mathcal{V}_h^D$	set of all vertices lying on $\bar{\Gamma}_D$ ,
$\mathcal{M}_h^D$	set of midpoints of edges lying on $\Gamma_D$ ,
$\mathcal{V}_h^N$	set of all vertices lying on $\Gamma_N$ ,
$\mathcal{M}_h^N$	set of midpoints of edges lying on $\Gamma_N$ ,
$\mathcal{V}_h^C$	set of all vertices lying on $\bar{\Gamma}_C$ ,
$\mathcal{M}_h^C$	set of midpoints of edges lying on $\Gamma_C$ ,
$\mathcal{V}_h^o$	$\mathcal{V}_h \setminus \mathcal{V}_h^D$ ,
$\mathcal{M}_h^o$	$\mathcal{M}_h \setminus \mathcal{M}_h^D$ ,
$\mathcal{E}_p$	set of all edges sharing the node $p$ ,
$\mathcal{T}_e$	set of all triangles sharing the edge $e$ ,
$\omega_p$	set of all elements sharing the node $p$ ,
$\gamma_{p,D}$	$\partial\omega_p \cap \Gamma_D$ ,
$\gamma_{p,N}$	$\partial\omega_p \cap \Gamma_N$ ,
$\gamma_{p,C}$	$\partial\omega_p \cap \Gamma_C$ ,
$\gamma_{p,I}$	set of all interior edges in $\omega_p$ ,
$h_p$	$\max\{h_T : T \in \omega_p\}$ ,
$h_s$	$\max\{h_e : e \in \gamma_p \text{ where } \gamma_p = \gamma_{p,I}, \gamma_{p,N} \text{ or } \gamma_{p,C}\}$ ,
$\mathbb{P}_k(T)$	the space of polynomials of degree atmost $k$ over element $T$ ,
$ S $	cardinality of the set $S$ ,
$X^*$	dual space of a Banach space $X$ ,
$L^p(\Omega)$	Lebesgue spaces, $1 \leq p \leq \infty$ ,
$\ \cdot\ _{L^p(D)}$	the associated norm on $L^p(D)$ where $D \subseteq \Omega$ ,
$(\cdot, \cdot)$	the associated inner product on $L^2(\Omega)$ ,
$W^{m,p}(\Omega)$	Sobolev spaces, $1 \leq p \leq \infty$ ,
$H^m(\Omega), H_0^m(\Omega)$	Sobolev spaces when $p = 2$

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$\ \cdot\ _{W^{m,p}(D)}$	the associated norm on $W^{m,p}(D)$ where $D \subseteq \Omega$ ,
$ \cdot _{W^{m,p}(D)}$	the associated semi norm on $W^{m,p}(D)$ where $D \subseteq \Omega$ ,
$\ \cdot\ _{H^m(D)}$	the associated norm on $H^m(D)$ where $D \subseteq \Omega$ ,
$ \cdot _{H^m(D)}$	the associated semi norm on $H^m(D)$ where $D \subseteq \Omega$ ,
$C^m(\Omega)$	the space of $m$ times continuously differentiable functions,
$C_c^m(\Omega)$	the space of $m$ times continuously differentiable functions with compact support in $\Omega$ ,
$D(\Omega)$	the space of infinitely differentiable functions having compact support in $\Omega$ ,
$\mathcal{I}$	time interval $(0, T]$ for some $T > 0$ ,
$L^p(\mathcal{I}; X)$	Bochner spaces, $1 \leq p \leq \infty$ ,
$W^{m,p}(\mathcal{I}; X)$	Bochner spaces, $1 \leq p \leq \infty$ ,
$C(\bar{\mathcal{I}}; X)$	space of continuous functions $v : \bar{\mathcal{I}} \rightarrow X$ ,
$\ \cdot\ _{L^p(\mathcal{I}; X)}$	the associated norm on $L^p(\mathcal{I}; X)$ ,
$\ \cdot\ _{W^{m,p}(\mathcal{I}; X)}$	the associated norm on $W^{m,p}(\mathcal{I}; X)$ ,
$\delta_{i,j}$	the Kronecker's Delta i.e., $\delta_{i,j} = 1$ if $i = j$ and 0 elsewhere,
$\mathbf{v}'$	the transpose of vector-valued function $\mathbf{v}$ ,
$C$	generic positive constant, independent of perturbation and mesh parameters,
$X \lesssim Y$	$X \leq CY$ for some positive constant $C$ , which is independent of the mesh size $h$ ,
$X \sim Y$	$C_1Y \leq X \leq C_2Y$ for some generic positive constants $C_1, C_2$ ,
$\text{supp}(f)$	support of function $f$ ,
$d$	dimension of the space.

