

NUMERICAL APPROXIMATION OF P AND P(X)-TYPE EXPONENT PROBLEMS USING WEB-SPLINE MESH-FREE METHOD

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by

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Dedicated to
My Gurus and My Family

Certificate

This is to certify that the thesis entitled “**Numerical Approximation of p and $p(x)$ -Type Exponent Problems Using WEB-Spline Mesh-free Method**” submitted by “**Mr. Naraveni Rajashekar**” to the Indian Institute of Technology Delhi, for the award of the Degree of Doctor of Philosophy, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

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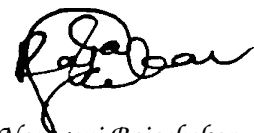
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Abstract

Partial differential equations have many important applications in various fields of science and engineering, such as fluid dynamics, materials science, inverse problems, computer graphics, and pattern formation. In recent years, increasing attention has been paid to the study of partial differential equations involving exponents p and $p(\mathbf{x})$ (variable exponent). The interest in studying such problems was stimulated by their applications in fluid dynamics, turbulent flows, non-Newtonian flows, climatology, elasticity, image processing, and electrorheological fluids. In this thesis, our aim is to find numerical solutions of $p(\mathbf{x})$ -Laplacian problem, $p(\mathbf{x})$ -Laplacian eigenvalue problem, p -biharmonic problem, and parabolic p -Laplacian problem. Below, we provide a few details of the above works. We propose minimization techniques (quasi-Newton and steepest descent) to solve the $p(\mathbf{x})$ -Laplacian problem using WEB-Spline method which is the generalization of Laplacian operator ($p(\mathbf{x}) = 2$) and p -Laplacian operator ($p(\mathbf{x}) = p$). We perform several numerical experiments to validate the results given in the literature.

Further, we analyze the numerical solution for computing the first eigenpair of $p(\mathbf{x})$ -Laplacian problem using WEB-Spline method. We adopt the inverse power method for computing the first eigenpair of $p(\mathbf{x})$ -Laplacian operator and also we study the asymptotic behaviour of the first eigenpair. Numerical simulations are performed to show the efficiency of the proposed scheme.

Next, we study the linear p -biharmonic problem (i.e., $p = 2$ case) with Navier boundary conditions. For solving the fourth order biharmonic problem, usually we need C^1 finite elements which are difficult to construct. Hence, we decoupled this fourth order problem

into two second order equations for which we can use C^0 finite elements. For this decoupled problem, we use WEB-Spline method and verify the theoretical results via numerical experiments.

Next, we extend WEB-Spline method from linear biharmonic problem to nonlinear p -biharmonic problem. For this problem, we perform numerical experiments to substantiate the theoretical results for linear case, and for nonlinear case no theoretical results are available; hence we gauge the rate of convergence.

Finally, we extended the WEB-Spline method from heat and p -Laplacian problems to a parabolic p -Laplacian problem. For time discretization, we use backward Euler method and for this fully-discrete formulation, we use Newton's method to linearize the nonlinear problem. We give a priori error estimates for semi-discrete and fully-discrete formulations. Also, we present a few examples to confirm the theoretical estimates.

आंशिक अंतर समीकरणों का विज्ञान और इंजीनियरिंग के विभिन्न क्षेत्रों में कई महत्वपूर्ण अनुप्रयोग हैं, जैसे कि द्रव गतिकी, सामग्री विज्ञान, व्युत्क्रम समस्याएं, कंप्यूटर ग्राफिक्स और पैटर्न गठन। हाल के वर्षों में, प्रतिघातों पी और पी (एक्स) (चर घातांक) से जुड़े आंशिक अंतर समीकरणों के अध्ययन पर अधिक ध्यान दिया गया है। इस तरह की समस्याओं का अध्ययन करने में रुचि, द्रव गतिशीलता, अशांत प्रवाह, गैर-न्यूटोनियन प्रवाह, जलवायु विज्ञान, लोच, छवि प्रसंस्करण और विद्युत-तरल पदार्थों में उनके अनुप्रयोगों से प्रेरित थी। इस थीसिस में हमारा उद्देश्य पी (एक्स) - लैप्लासियन समस्या, पी (एक्स) - लैप्लासियन आइजन वैल्यू समस्या, पी-बायहार्मोनिक समस्या और पैराबोलिक पी-लैप्लासियन समस्या के संख्यात्मक समाधान ढूंढना है। नीचे, हम उपरोक्त कार्यों के कुछ विवरण प्रदान करते हैं। हम वेब-स्प्लीन विधि का उपयोग करके पी (एक्स) - लैप्लासियन समस्या को हल करने के लिए न्यूनतमीकरण तकनीकों (अर्ध-न्यूटन और सबसे तेज वंश) का प्रस्ताव करते हैं जो लैप्लासियन ऑपरेटर (पी (एक्स) = 2) और पी-लैप्लासियाई ऑपरेटर (पी(एक्स) = पी) का सामान्यीकरण है। हम साहित्य में दिए गए परिणामों को मान्य करने के लिए कई संख्यात्मक प्रयोग करते हैं।

इसके अलावा, हम वेब-स्प्लीन विधि का उपयोग करके पी (एक्स) के पहले आइजनपेयर की गणना के लिए संख्यात्मक समाधान का विश्लेषण करते हैं। हम पी (एक्स) के पहले आइजनपेयर की गणना के लिए विलोम शक्ति विधि अपनाते हैं और हम पहले आइजनपेयर के स्पर्शोन्मुख व्यवहार का भी अध्ययन करते हैं। प्रस्तावित योजना की दक्षता दिखाने के लिए संख्यात्मक सिमुलेशन किए जाते हैं।

इसके बाद, हम नवियर सीमा की स्थितियों के साथ रैखिक पी-बायहार्मोनिक समस्या (यानी, पी = 2 केस) का अध्ययन करते हैं। चौथे क्रम की बायहार्मोनिक समस्या को हल करने के लिए, आमतौर पर हमें C_1 परिमित तत्वों की आवश्यकता होती है, जिनका निर्माण मुश्किल होता है। इसलिए, हमने इस चौथे क्रम की समस्या को दो दूसरे क्रम समीकरणों में बदल दिया, जिसके लिए हम C_0 परिमित तत्वों का उपयोग कर सकते हैं। इस अवनत समस्या के लिए, हम वेब-स्प्लीन विधि का उपयोग करते हैं और संख्यात्मक प्रयोगों के माध्यम से सैद्धांतिक परिणामों को सत्यापित करते हैं।

इसके बाद, हम रैखिक बायहार्मोनिक समस्या से गैर-पी-बायार्मोनिक समस्या के लिए वेब-स्प्लीन विधि का विस्तार करते हैं। इस समस्या के लिए, हम रैखिक मामले के लिए सैद्धांतिक परिणामों को प्रमाणित करने के लिए संख्यात्मक प्रयोग करते हैं, और गैर-रैखिक मामले के लिए कोई सैद्धांतिक परिणाम उपलब्ध नहीं है; इसलिए हम अभिसरण की दर को नापते हैं।

अंत में, हमने हीट और पी-लैप्लासियन समस्याओं को वेब-स्प्लीन विधि से पैराबोलिक पी-लैप्लासियन समस्या तक बढ़ाया है। समय सावधानी के लिए, हम पिछड़े यूलर विधि का उपयोग करते हैं और इस पूरी तरह से असतत निर्माण के लिए, हम गैर-रैखिक समस्या को रैखिक करने के लिए न्यूटन की विधि का उपयोग करते हैं। हम अर्ध-असतत और पूरी तरह से असतत योगों के लिए एक प्राथमिकताओं त्रुटि अनुमान देते हैं। इसके अलावा, हम सैद्धांतिक अनुमानों की पुष्टि करने के लिए कुछ उदाहरण प्रस्तुत करते हैं।

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List of Symbols

Symbol	Meaning
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
$\forall x$	For all x
$x \in X$	x is a member of X
Δ	Laplacian
$\preceq, \succeq$	Inequalities upto constants
∇	Gradient
M'	Dual Space of M
$\mathbf{B}^T$	Transpose of Matrix or Vector $\mathbf{B}$
I	Set of Inner nodes
J	Set of Outer nodes
$\delta_{i,k}$	Kronecker delta
Λ_k	Dual functional of WEB-Spline B_k
$\mathcal{P}_h$	Canonical projection operator
$\mathbb{R}^d$	d -dimensional real Euclidean space
∂U	Boundary of U

$Y \subset Z$	Y is a subset of Z
$supp$	Support of a function
div	Divergence
$L^2(\Omega)$	Square integrable function
$H_0^k(\Omega)$	Trace zero elements of Sobolev space $H^k(\Omega)$