# Functions between metric spaces and their relations to cofinal completeness

**Lipsy**

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
July 2021

# Functions between metric spaces and their relations to cofinal completeness


by

**Lipsy**

Department of Mathematics


*Submitted*

*in fulfillment of the requirements of the degree of Doctor of Philosophy*

*to the*


**Indian Institute of Technology Delhi**


**July 2021**

*To My Parents*

*Smt. Neelam Devi*

*Sh. Parmod Kumar*

# Certificate

This is to certify that the thesis entitled **Functions between metric spaces and their relations to cofinal completeness** submitted by **Miss Lipsy** to the Indian Institute of Technology Delhi, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bona fide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
DATE

**Dr. Subiman Kundu**
**Professor**
**Department of Mathematics**
**Indian Institute of Technology Delhi**

# Acknowledgements

First I want to express my deep respect and heartfelt gratitude to my thesis advisor Prof. Subiman Kundu. I have been fortunate to be advised by him. His honesty, never-ending passion for teaching, and the joy that he takes in understanding things deeply inspire me a lot. I am thankful for his patience, for all his academic advice, and his valuable feedback on all my queries, which helped to improve the quality of the thesis. I truly appreciate the freedom he gave me with my work and working hours. There's a famous quote by Maimonides -"Give a man a fish and you feed him for a day; teach a man to fish and you feed him for a lifetime". His guidance approach has helped me to grow mathematically.

I take this opportunity to thank all the instructors of the courses that I have done throughout my mathematical journey. I am deeply indebted to Prof. Sivananthan Sampath for teaching Real Analysis in the first semester of my master's. That particular course laid the foundation of my interest in analysis which was further strengthened by various courses taught by Prof. Kundu. I would also like to thank my master's thesis advisor Prof. Ritumoni Sarma for his encouragement.

I am indebted to Dr. Manisha Aggarwal for her invaluable suggestions and insightful guidance during my research. I would also like to thank my committee members - Prof. S. D. Joshi, Prof. K. Sreenadh, and Prof. Shravan Kumar for their time and encouragement. I acknowledge IIT Delhi for providing me GATE fellowship and travel grants during my Ph.D. I appreciate IIT Delhi authorities, hostel staff, and security for providing a safe environment.

I would like to thank all the amazing people who have made my Ph.D. life enjoyable and colorful. Special thanks to my friends Lalit, Ruchika, Juhi, Sakshi, Anuj, Navnit and Dileep for making my stay at IIT memorable and meaningful. I can't thank my friend Navnit Yadav enough for always believing in me, for always inspiring me to dream big and for challenging me every day to be the best that I can be! I admire Lalit's incredible friendship and thank Ruchika for her quick bits of help and all the parties at her place. I am also thankful to Bhawna, Kapil, Anisha, Tanvi, Jyotsna, Ritesh, Priyamvada, Pooja, Vaibhav, Hariom, Sandeep, Abhay, Saurabh, Abhilash, Ankita Jindal, Ratan, Manpreet and Manoj for our times together. A big thanks go to my lovely roommates Vasudha Khubchandani and Meghashree Padhan for their love and for all the delicious food they cook. I deeply appreciate all the efforts made by Navnit, Indresh and Lalit to teach me lawn tennis. I would like to thank Mrs. Pravita Singh for her kind and inspiring nature, and her warm welcome whenever I visited her place in IIT. Outside IIT, I am lucky to count my childhood friends - Isha and Aman. I thank Isha for being such an understanding, caring, kind and irreplaceable friend. Irrespective of my mood, Aman never fails to make me laugh. I thank them for taking me out of the IIT bubble from time to time. I duly acknowledge my friend Manisha Garg for our deep discussions, for her honest life advice, and for always being there in moments of need. I am grateful to her for showing belief in my potential and for always inspiring me to learn mathematics with pure love and intentions. I owe a lot to my friends from my master's - Yogita, Sonam, Sarita, Vijay, Priyal, and Mohit for their love and support. I am also thankful to Arpit Goyal for his friendship.

Last, but foremost, I express my deepest sense of gratitude to my parents and my family for their patience, unconditional love and endless support. From the very beginning, my education has been very important for my parents. I am thankful to my father and my late grandfather Sh. Om Parkash Garg for teaching me since the early days of my childhood. My mother always inspires me to stay positive in every situation. I am indebted to her for suggesting me to pursue a career in mathematics. I am also grateful to my younger brother Vivek Garg for his love, support and best wishes. I express my sincere thanks to all my loving family members.

I thank everyone who interacted with me and shared me their knowledge and kindness.

New Delhi Lipsy

July 2021

# Abstract

This thesis is devoted to the study of various families of functions between metric spaces. A function between two metric spaces is said to be Cauchy-regular if it preserves Cauchy sequences. In the literature, we have various weaker forms of Cauchy sequences such as cofinally Cauchy sequences, pseudo-Cauchy sequences, Bourbaki-Cauchy sequences, and cofinally Bourbaki-Cauchy sequences. The corresponding spaces in which these sequences cluster are called cofinally complete, UC, Bourbaki-complete, and cofinally Bourbaki-complete respectively. All these spaces lie strictly in between the class of compact metric spaces and that of complete metric spaces.

Similar to the idea of the well-studied class of Cauchy-regular functions, we have functions that preserve the aforementioned classes of weaker Cauchy sequences known as CC-regular, PC-regular, BC-regular, and CBC-regular functions. Along with these functions, various properties of other functions such as almost bounded, Cauchy-subregular, Cauchy-regular, uniformly continuous, strongly uniformly continuous, and some Lipschitz-type functions are also investigated. This study leads to various characterizations of cofinally complete metric spaces and the spaces that are stronger than the cofinally complete ones that are UC spaces and cofinally Bourbaki-complete spaces. Furthermore, cofinally complete metric space are characterized in terms of some function space topologies as well.

# सार

यह थीसिस मीट्रिक स्थानों के बीच फलनों के विभिन्न परिवारों के अध्ययन के लिए समर्पित है। दो मीट्रिक स्थानों के बीच एक फलन को कॉची-नियमित कहा जाता है यदि यह कॉची अनुक्रमों को संरक्षित करता है। साहित्य में, हमारे पास कॉची अनुक्रमों के विभिन्न कमजोर रूप हैं जैसे कि कोफ़ाइनली कॉची अनुक्रम, स्यूडो-कॉची अनुक्रम, बॉर्बकी-कॉची अनुक्रम, और कोफ़ाइनली बॉर्बकी-कॉची अनुक्रम। संबंधित स्थानों, जिसमें ये अनुक्रम क्लस्टर होती हैं, उनको कोफ़ाइनली कम्प्लीट, यूसी, बॉर्बकी-कम्प्लीट, और कोफ़ाइनली बॉर्बकी-कम्प्लीट कहा जाता है। ये सभी स्थान कॉम्पैक्ट मीट्रिक स्थान और कम्प्लीट मीट्रिक स्थान के वर्ग के बीच सख्ती से स्थित हैं।

कॉची-नियमित फलन के अच्छी तरह से अध्ययन किए गए वर्ग के विचार के समान, हमारे पास ऐसे फलन हैं जो सीसी-नियमित, पीसी-नियमित, बीसी-नियमित और सीबीसी-नियमित फलनों के रूप में जाने जाने वाले कमजोर कॉची अनुक्रमों के उपरोक्त वर्गों को संरक्षित करते हैं। इन फलनों के साथ, अन्य फलन, जैसे कि लगभग बाध्य, कॉची-उपनियमित, कॉची-नियमित, समान रूप से निरंतर, दृढ़ता से समान रूप से निरंतर, और कुछ लिप्सचिट्ज़-प्रकार के फलनों के विभिन्न गुणों की भी जांच की गई है। इस अध्ययन से कोफ़ाइनली कम्प्लीट मीट्रिक स्थान और स्थान जो कोफ़ाइनली कम्प्लीट मीट्रिक स्थानों से अधिक मजबूत हैं, जैसे कि यूसी स्थान और कोफ़ाइनली बॉर्बकी-कम्प्लीट स्थान, के विभिन्न चित्रण पाए गए हैं। इसके अलावा, कुछ फलन स्पेस टोपोलोजीस के संदर्भ में भी कोफ़ाइनली कम्प्लीट मीट्रिक स्थानों को चित्रित किया गया है।

# Contents

# List of Symbols

| Symbol | Meaning |
|---|---|
| $\forall$ | for all |
| $\exists$ | there exists |
| $\in$ | belongs to |
| $\notin$ | does not belong to |
| $\subseteq$ | subset or equal |
| $\cup$, $\cap$ | union, intersection |
| $X \setminus E$ or $E^c$ | the complement of $E$ in $X$ |
| $\emptyset$ | empty set |
| $\mathbb{N}$ | the set of natural numbers |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{R}$ | the real line |
| $l_2$ | the Hilbert space of square summable real sequences |
| $(e_n)$ | the standard orthonormal basis in $l_2$ |
| $\square$ | end of a proof |

| | |
|---|---|
| $\lvert\cdot\rvert$ | the usual distance metric on $\mathbb{R}$ |
| $\lVert\cdot\rVert$ | norm |
| $(x_n)_{n\in\mathbb{N}}$ | a sequence in a non-empty set, occasionally it may be denoted by $(x_n)$ |
| $f\vert_A$ | the restriction of $f$ to $A$ where $f: X \to Y$ is a function, $\emptyset \neq A \subseteq X$ |

---

For the following notations, $(X, d)$ is a metric space, $A$ is a non-empty subset of $X$, $x \in X$, and $\epsilon > 0$

---

| | |
|---|---|
| $\overline{A}$ or $cl_X A$ | the closure of $A$ in $X$ |
| $intA$ or $A^\circ$ | the interior of $A$ in $X$ |
| $X'$ | the set of accumulation points in $X$ |
| $(\widehat{X}, d)$ | the completion of a metric space $(X, d)$ |
| $B_d(x,\epsilon)$ or $B(x,\epsilon)$ or $B^1(x,\epsilon)$ | the open ball in $(X, d)$, centered at $x \in X$ with radius $\epsilon$ |
| $C_d(x,\epsilon)$ or $C(x,\epsilon)$ | the closed ball in $(X, d)$, centered at $x \in X$ with radius $\epsilon$ |
| $d(x, A)$ | distance between $x$ and $A$ which is $\inf\{d(x,a) : a \in A\}$ |
| $A^\epsilon$ or $B(A,\epsilon)$ | the $\epsilon$-enlargement of $A$, that is, the set $\{x \in X : d(x, A) < \epsilon\}$ |
| for every $n \geq 2$, $B^n(x,\epsilon)$ | the $\epsilon$-enlargement of the set $B^{n-1}(x,\epsilon)$ |

# Introduction

Functions play an important role in the theory of metric spaces, more precisely in analysis on metric spaces. Two important classes of functions, namely the class of continuous functions and that of uniformly continuous functions are well-known to all of us. In [79], Snipes has studied the functions that lie strictly in between these two important classes of functions, which he called Cauchy-regular functions. A function between two metric spaces is said to be Cauchy-regular if it preserves Cauchy sequences, that is, it takes Cauchy sequences to Cauchy sequences. These functions were further investigated by Snipes and Borsík in [80, 24, 25]. Since Cauchy-regular functions are continuous, they have also been widely known as Cauchy-continuous functions in the literature [1, 17, 18]. It is well-known that every continuous function on a metric space $(X, d)$ is Cauchy-regular if and only if $(X, d)$ is complete, whereas every Cauchy-regular function on $(X, d)$ is uniformly continuous if and only if the completion $(\widehat{X}, d)$ is a UC space [7]. Cauchy-regular functions play a vital role in the analysis of complete metric spaces as well as the spaces which are stronger than the complete ones. Recall that a metric space is compact if and only if it is totally bounded and complete. Thus there is a huge gap between compact metric spaces and complete metric spaces. The role of these two classes of metric spaces in mathematical analysis is beyond question. Thus many researchers have been inspired to study the metric spaces which lie strictly in between these two classes of metric spaces. The classes of cofinally complete metric spaces, UC spaces, Bourbaki-complete metric spaces, cofinally Bourbaki-complete metric spaces are some better-known examples of such spaces. Let us first discuss various weaker forms of Cauchy sequences which define or characterize these spaces and then we will discuss the functions that preserve such sequences.

Recall that a sequence $(x_n)$ is Cauchy if for every $\epsilon > 0$, there exists a residual set of indices $\mathbb{N}_\epsilon$ such that each pair of terms, the indices of which, come from $\mathbb{N}_\epsilon$ are within $\epsilon$ distance of each other. If we replace "residual" by "cofinal" then we obtain sequences that are called cofinally Cauchy. *A metric space is called cofinally complete if each cofinally Cauchy sequence in it has a cluster point.* We all know that every continuous function from a compact metric space to an arbitrary metric space is uniformly continuous. But this property does not characterize compactness. It is a characteristic property of a larger class of metric spaces called UC spaces (also widely known as Atsuji Spaces). *A metric space is said to be UC if every real-valued continuous function on it is uniformly continuous.* The corresponding weaker form of Cauchy sequence used to characterize UC spaces is pseudo-Cauchy sequence. The notion of pseudo-Cauchy sequence was introduced by Toader in [82]. More precisely, a metric space is UC if and only if each pseudo-Cauchy sequence in it with distinct terms has a cluster point, where a sequence is called pseudo-Cauchy if there exists a pair of terms arbitrarily close frequently. Since every cofinally Cauchy sequence is pseudo-Cauchy, every UC space is cofinally complete. In 1958, Atsuji [4] introduced the notion of finitely chainable metric spaces in order to characterize the metric spaces on which every real-valued uniformly continuous function is bounded. Atsuji proved: every real-valued uniformly continuous function on a metric space is bounded if and only if the metric space is finitely chainable. The class of finitely chainable metric spaces strictly lies in between the class of bounded metric spaces and that of totally bounded metric spaces (see also [60]). Finitely chainable metric spaces are also called Bourbaki bounded in the literature because these sets were considered in the book of Bourbaki [26]. To characterize finite chainability sequentially, Garrido and Meroño defined Bourbaki-Cauchy and cofinally Bourbaki-Cauchy sequences [41]. Corresponding to these sequences, Garrido and Meroño have introduced two new types of complete metric spaces, namely Bourbaki-complete metric spaces and cofinally Bourbaki-complete metric spaces. A metric space is said to be (cofinally) Bourbaki-complete if every (cofinally) Bourbaki-Cauchy sequence in it clusters. The concept of cofinally Bourbaki-Cauchy sequences is similar to that of cofinally Cauchy sequences, in fact, every cofinally Bourbaki-complete metric space is cofinally complete. There is no direct relation between cofinally complete metric spaces and Bourbaki-complete metric spaces, but interestingly, the collection of cofinally Bourbaki-complete metric spaces is positioned in between the class of UC spaces and that of cofinally complete metric spaces.

In the literature, one can find a comprehensive list of characterizations of the aforementioned variants of complete metric spaces in terms of geometric functionals, certain sequences and functions, Cantor-type conditions, etc. In [15, 16, 17, 18, 19, 37], various authors have considered some Lipschtiz-type functions such as locally Lipschitz, Cauchy-Lipschitz, uniformly locally Lipschitz and Lipschitz in the small. In these articles, various interesting properties of these functions have been investigated. In fact, complete metric spaces and their variants have also been characterized in terms of these Lipschitz-type functions in context of their relation with each other. Also, having been motivated by the significance of Cauchy-regular functions and complete metric spaces, in 2016, Aggarwal and Kundu [2] have defined the functions that preserve various weaker forms of Cauchy sequences that we have discussed, and characterized the corresponding classes of complete metric spaces using those functions. The functions that preserve cofinally Cauchy sequences are called CC-regular functions, whereas the functions that preserve pseudo-Cauchy sequences are called PC-regular functions. Similarly, the functions that preserve (cofinally) Bourbaki-Cauchy sequences are called (CBC-regular) BC-regular functions. The primary objective of the thesis is to investigate various interesting properties of the functions that have been talked about so far. We also present some analysis on strongly uniformly continuous functions which were first defined in [21] and

Cauchy-subregular functions, where a function is said to be Cauchy-subregular if takes Cauchy sequences to sequences having a Cauchy subsequence. Some relations among not necessarily continuous functions such as CC-regular, PC-regular, Cauchy-subregular along with their relations to continuous, Cauchy-regular, uniformly continuous and Lipschitz-type functions in various combinations have been established. As consequences, we obtain nice characterizations of cofinally complete metric spaces and the spaces which are stronger than cofinally complete spaces, that is, UC spaces and cofinally Bourbaki-complete spaces. The study of boundedness of various combinations of Lipschitz-type functions and other functions gives us characterizations of totally bounded and finitely chainable metric spaces. We also give characterizations of cofinally complete metric spaces in terms of some function space topologies. Before going into the details of the organisation of the thesis, we would like to give a brief historical review on cofinally complete metric spaces and UC spaces.

Out of the four variants of complete metric spaces that we have talked about, the concept of UC spaces is the oldest one. According to [27], the study of UC spaces can be traced back to at least 1947 [33], if not earlier. Then Nagata studied such spaces in 1950 in [69], followed by Monteiro and Peixoto [66] in 1951. The UC spaces were first extensively studied by Atsuji [4] in 1958 and hence they have widely been known as Atsuji spaces also in the literature. The UC spaces have been the subject of study for a number of articles over the decades [6, 7, 8, 30, 73, 83]. A wide collection of equivalent conditions for a metric space to be a UC space can be found in the survey article [61] of Kundu and Jain. Cofinal completeness was first considered implicitly by Corson [31] in 1958 and then by Howes [51] in 1971 in terms of nets and entourages. A few years later, Rice [74] introduced the notion of uniform paracompactness for a Hausdorff uniform space $X$ and subsequently in [78], Smith, the reviewer of Rice's paper for Mathematical Reviews, observed that uniform paracompactness is equivalent to net cofinal completeness for a Hausdorff uniform space. In the context of metric spaces, a metric space $(X, d)$ is called uniformly paracompact if for each open cover $\mathcal{V}$ of X, there exists an open refinement $\mathcal{U}$ and $\delta > 0$ such that for each $x \in X$, $B(x, \delta)$ intersects only finitely many members of $\mathcal{U}$. In [13], it has been shown that sequential cofinal completeness in metric spaces is equivalent to uniform paracompactness. In 1981, Hohti [47] gave a nice equivalent characterization of a uniformly paracompact metric space in terms of uniform local compactness. Much later in 2008, Beer [10] cast a new light on cofinal completeness and gave various nice characterizations of cofinally complete metric spaces, presenting unanticipated parallels between this class of metric spaces and UC spaces. Later on, in 2016, Aggarwal and Kundu added various characterizations of UC spaces and cofinally complete metric spaces [1, 2].

The entire work of the thesis has been presented in five chapters.

In Chapter 1, we define the variants of complete metric spaces which we have mentioned earlier and various examples are given to illustrate the relation of these spaces with each other. Moreover, various Lipschitz-type functions, strongly uniformly continuous functions, Cauchy-subregular functions, functions preserving certain sequences, are briefly introduced. The connections of these functions with the variants of complete metric spaces, which have already been established in the literature are also discussed. Some properties of Lipschitz-type functions and some extension results related to continuous, Cauchy-regular, and uniformly continuous functions are mentioned as well. Since the thesis is a bit long, we have avoided giving proofs of the results given in this chapter.

In Chapter 2, we focus on CC-regular functions and their relations with other functions. In 2017, Keremedis has defined almost bounded functions and AUC spaces [56]. We show that an almost bounded function is nothing but a CC-regular function and an AUC space is nothing but a cofinally complete metric space. We study stability of never zero continuous CC-regular functions under reciprocation and boundedness of various combinations of Lipschitz-type functions with CC-regular functions. Further, we compare CC-regular functions with Cauchy-subregular functions. In the entire process, we obtain characterizations of cofinally complete metric spaces and the spaces which have cofinal completion. We also explore the condition under which a Cauchy-subregular function is Cauchy-regular.

In Chapter 3, we characterize UC spaces in terms of uniform continuity of various thin subclasses of continuous functions. The comparison of Cauchy-subregular functions and PC-regular functions leads to various characterizations of metric spaces having UC completion. Conditions under which PC-regular functions, strongly uniformly continuous functions, CBC-regular functions are stable under reciprocation, are also studied. Investigation on some properties of CBC-regular functions gives us characterizations of cofinally Bourbaki-complete metric spaces and finitely chainable spaces.

In Chapter 4, we study metric spaces with cofinally complete completion; for brevity, we say that such spaces have cofinal completion. Since cofinally complete metric spaces are complete, it is natural to pay attention towards the metric spaces which have cofinal completions, as some metric spaces are deprived of being cofinally complete just because they are not complete (same goes with any other variant of complete metric spaces). We study such spaces in terms of some properties like boundedness, uniform continuity of Cauchy-regular functions, Cauchy-Lipschitz function, and uniformly locally Lipschitz functions defined on them. One of the key tools that we use is the local total boundedness functional which measures the local total boundedness of a metric space at its each point.

In Chapter 5, we characterize cofinally complete metric spaces in terms of some function space topologies. It is well-known that if a metric space $(X, d)$ is compact and $(Y, \rho)$ is any other metric space, then on the set of continuous functions from $X$ to $Y$, the topology of uniform convergence is same as the topology given by the Hausdorff metric [70, Theorem 4.7]. This equivalence plays a significant role in approximation theory, for instance see [77]. However, in general, one may need to deal with non-compact metric spaces. Thus we establish the same equivalence on some particular subsets of continuous functions and locally Lipschitz functions from $(X, d)$ to $(Y, \rho)$, where the equivalence also characterizes the cofinal completeness of the space $(X, d)$.

Unless mentioned otherwise, $\mathbb{R}$ and its non-empty subsets carry the usual distance metric and all metric spaces are considered to be infinite. Finally, in the thesis, we take one numbering for the Definitions, one for the Examples, one for the Remarks and another one for the Propositions, Lemmas, Theorems and Corollaries, each numbering being restricted to its own chapter.