

**ANALYSIS AND DESIGN OF TIME-DELAYED  
SYSTEMS: A LAMBERT W FUNCTION APPROACH**

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# ANALYSIS AND DESIGN OF TIME-DELAYED SYSTEMS: A LAMBERT W FUNCTION APPROACH

by

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# CERTIFICATE

This is to certify that the thesis entitled “**Analysis and design of time-delayed systems: A Lambert W function approach**”, submitted by **Niraj Choudhary** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** in Electrical Engineering, is a record of the original, bonafide research work carried out by her under our supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations related to the award of the degree.

The results contained in this thesis have not been submitted either in part or in full to any other university or institute for the award of any degree or diploma to the best of our knowledge.

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# ABSTRACT

Most practical systems have inherent time delay by virtue of the sensors, actuators and interaction between its various components. Time delay is usually a cause of instability and poor performance and significantly increases the complexity in analysis and design of such systems. Consequently, the study of stability and control of these time delay systems (TDSs) is of significant theoretical and practical importance.

Time delay systems involve a transcendental characteristic equation that has infinite eigenvalues. The conventional methods for delay-free systems can not be employed directly for analyzing TDSs due to their transcendental behavior. As an alternative, the concept of the Lambert W function has been explored in this thesis to facilitate the system analysis and controller design for TDSs. The Lambert W function based method yields an analytical solution of TDSs that is analogous to that of ordinary differential equations. The analogy between the solution forms of the delay differential equation and the ordinary differential equation aids in the extension of conventional approaches for delay-free systems to systems with time delay. This thesis proposes various control algorithms for a different class of linear time-invariant TDSs based on the Lambert W function.

The matrix Lambert W function framework has been utilized in this thesis to investigate the eigenspectrum of the higher order TDSs. This investigation leads to the development of an algorithm that circumvents the discrepancies and limitations of the state-of-art approach. The concept of the Lambert W function for solving multidimensional TDSs with multiple discrete delays has also been explored in this thesis.

Subsequently, the control design problem for such systems has been addressed through the Lambert W function based approach. Uncertainties and external disturbances are practical phenomena, often experienced by dynamical systems. Sliding mode control approach has been chosen to address such system uncertainties because of its two magnetic properties; first is

the order reduction, and the second one is the invariance property (insensitive to matched uncertainties). In this context, the Lambert W function based approach is utilized to design the sliding manifold, which assures the stability of the reduced order system.

As full state measurement is not possible in many practical cases, the work is extended to the design of an observer-based controller for the artificially induced time delay systems. In order to accomplish this, the thesis proposes a composite design of a state estimator appended with a disturbance observer for uncertain input delay systems. The case of state delay systems with bounded uncertainty have also be handled in this context via sliding mode control.

Furthermore, dynamical systems, like diffusion processes, heat transfer problem, etc. cannot be represented precisely using integer order differential equations. Hence, fractional differential equations are utilized to model such systems, and the corresponding systems are referred as 'fractional order systems'. The study of these systems is a challenging task; besides, the presence of delay contribute to the increased complexity in their analysis. The Lambert W function concept has also been used to bring about sliding mode control in the framework of fractional order systems. The unavailability of states has also been handled in this context by proposing an observer-based control algorithm.

Overall, this thesis proposes a Lambert W function based design framework for the analysis and design of time-delayed systems. The proposed methodologies not only provide an alternative to the conventional Lyapunov based techniques but also bring new insights on the recently established Lambert W function based methodology.

**Keywords:** Time delay systems; Lambert W function; Sliding mode control.

# सार

अधिकांश व्यावहारिक प्रणालियों में अंतर्निहित देरी सेंसर, एक्ट्यूएटर और विभिन्न घटकों के बीच इंटरैक्शन के आधार पर होती है। समय की देरी आमतौर पर अस्थिरता और खराब प्रदर्शन का कारण है और इस तरह सिस्टम के विश्लेषण और डिजाइन में जटिलता बढ़ जाती है। नतीजतन, इन समय-देरी प्रणालियों (टीडीएस) की स्थिरता और नियंत्रण का अध्ययन सैद्धांतिक और व्यावहारिक तरीके से महत्वपूर्ण है।

समय की देरी प्रणालियों में एक ट्रांसिजेंट विशेषता समीकरण होता है जिसमें अनंत गेनवाल्यूएस होते हैं। विलंब-मुक्त प्रणालियों के लिए, पारंपरिक तरीकों को उनके ट्रांसिजेंट व्यवहार के कारण सीधे टीडीएस के विश्लेषण के लिए नियोजित नहीं किया जा सकता है। एक विकल्प के रूप में, TDSs के लिए सिस्टम विश्लेषण और नियंत्रक डिजाइन की सुविधा के लिए इस थीसिस में लैम्बर्ट डब्ल्यू फ़ंक्शन की अवधारणा का पता लगाया गया है। लैम्बर्ट डब्ल्यू फ़ंक्शन आधारित पद्धति टीडीएस का एक विश्लेषणात्मक समाधान निकालती है जो साधारण अंतर समीकरणों के अनुरूप है। देरी अंतर समीकरण और साधारण अंतर समीकरण के समाधान रूपों के बीच समानता, विलम्ब से मुक्त सिस्टम के लिए पारंपरिक दृष्टिकोण के विस्तार को प्रस्तावित करती है। यह थीसिस लैम्बर्ट डब्ल्यू फ़ंक्शन के आधार पर रैखिक टाइम-वैरिएबल टीडीएस के एक अलग वर्ग के लिए विभिन्न नियंत्रण एल्गोरिदम का प्रस्ताव करता है।

मैट्रिक्स लैम्बर्ट डब्ल्यू फ़ंक्शन फ्रेमवर्क का उपयोग इस थीसिस में उच्च क्रम के टीडीएस के ईगेसपेक्टम की जांच करने के लिए किया गया है। यह जांच एक एल्गोरिथम के विकास की ओर ले जाती है जो अत्याधुनिक दृष्टिकोण की विसंगतियों और सीमाओं को दरकिनार करती है। बहुविकल्पीय टीडीएस को कई असतत देरी के साथ हल करने के लिए लैम्बर्ट डब्ल्यू फ़ंक्शन की अवधारणा को भी इस थीसिस में खोजा गया है।

इसके बाद, इस तरह के सिस्टम के लिए नियंत्रण डिजाइन समस्या को लैम्बर्ट डब्ल्यू फ़ंक्शन आधारित दृष्टिकोण के माध्यम से संबोधित किया गया है। अनिश्चितताओं और बाहरी गड़बड़ी व्यावहारिक घटनाएं हैं, जिन्हें अक्सर गतिशील प्रणालियों द्वारा अनुभव किया जाता है। दो महत्वपूर्ण विशेषताओं के कारण ऐसी प्रणाली अनिश्चितताओं को दूर करने के लिए स्लाइडिंग मोड कंट्रोल अप्रोच को चुना गया है; पहला ऑर्डर में कमी है, और दूसरा एक आक्रामक संपत्ति है (मिलान अनिश्चितताओं के प्रति असंवेदनशील)। इस संदर्भ में, लैम्बर्ट डब्ल्यू फ़ंक्शन आधारित दृष्टिकोण का उपयोग स्लाइडिंग मैनिफोल्ड को डिजाइन करने के लिए किया जाता है, जो सरल क्रम प्रणाली की स्थिरता का आश्वासन देता है।

जैसा कि कई व्यावहारिक मामलों में पूर्ण अवस्था माप संभव नहीं है, इसलिए यह काम कृत्रिम रूप से प्रेरित समय देरी प्रणालियों के लिए एक पर्यवेक्षक-आधारित नियंत्रक के डिजाइन तक विस्तारित है। इसे पूरा करने के लिए, थीसिस अनिश्चित स्थिति में देरी प्रणाली के लिए गड़बड़ी पर्यवेक्षक के साथ संलग्न एक स्थिति आकलनकर्ता के एक समग्र डिजाइन का प्रस्ताव है। बाउंडेड अनिश्चितता के साथ स्थिति देरी सिस्टम के मामले को भी इस संदर्भ में स्लाइडिंग मोड नियंत्रण के माध्यम से नियंत्रित किया जाता है।

इसके अलावा, डायनेमिक सिस्टम, जैसे डिफ्यूजन प्रोसेस, हीट ट्रांसफर प्रॉब्लम इत्यादि को पूर्णांक ऑर्डर डिफरेंशियल इक्वेशन का उपयोग करके सटीक रूप से प्रस्तुत नहीं किया जा सकता है। इसलिए, इस तरह के सिस्टम को मॉडल करने के लिए आंशिक भिन्न समीकरणों का उपयोग किया जाता है, और संबंधित

प्रणालियों को 'आंशिक आदेश प्रणाली' कहा जाता है। इन प्रणालियों का अध्ययन एक चुनौतीपूर्ण कार्य है; इसके अलावा, देरी की उपस्थिति उनके विश्लेषण में वृद्धि हुई जटिलता में योगदान करती है। लैम्बर्ट डब्ल्यू फ़ंक्शन अवधारणा का उपयोग आंशिक आदेश प्रणालियों के ढांचे में स्लाइडिंग मोड नियंत्रण को लाने के लिए भी किया गया है। एक पर्यवेक्षक आधारित नियंत्रण एल्गोरिथ्म का प्रस्ताव करके अवस्था की अनुपलब्धता को भी इस संदर्भ में नियंत्रित किया गया है।

कुल मिलाकर, यह थीसिस समय-विलंबित प्रणालियों के विश्लेषण और डिजाइन के लिए एक लैम्बर्ट डब्ल्यू फ़ंक्शन आधारित डिजाइन ढांचे का प्रस्ताव करती है। प्रस्तावित कार्यप्रणाली न केवल पारंपरिक लयापुणोव आधारित तकनीकों का एक विकल्प प्रदान करती है, बल्कि हाल ही में स्थापित लैम्बर्ट डब्ल्यू फ़ंक्शन आधारित कार्यप्रणाली पर नई अंतर्दृष्टि भी लाती है।

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# Nomenclature

## Acronyms

CC	Common Canonical
DDE	Delay Differential Equation
DOF	Degree of Freedom
DR	Dynamic Resonator
FC	Fractional Calculus
FDE	Functional Differential Equation
FO	Fractional Order
FOS	Fractional Order System
FOTDS	Fractional Order Time Delay System
LKF	Lyapunov-Krasovskii Functional
LMI	Linear Matrix Inequality
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
LWF	Lambert W Function
ODE	Ordinary Differential Equation
PI	Proportional Integrator
PID	Proportional Integral Derivative
QPmR	Quasi-Polynomial mapping based Root finder

## Acronyms

SF	State Feedback
SISO	Single Input Single Output
SMC	Sliding Mode Control
SSD	State Plus State Derivative
STM	State Transition Matrix
TDS	Time Delay System
TCE	Transcendental Characteristic Equation
UUB	Uniformly Ultimately Bounded

## List of Symbols

$(.)^T$	General notation for the matrix transpose operation
$\mathbb{C}$	Set of complex numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{Z}$	Set of integer numbers
$A$	State matrix in continuous-time system
$A_d$	State matrix in continuous-time system with time delay
$B$	Input matrix of control in continuous-time system
$C$	Output matrix in continuous-time system
$e$	Error between system and reference state vectors
$E_\alpha(\cdot)$	Mittag-Leffler function

## List of Symbols

$E_{\alpha,\beta}(\cdot)$	Generalized Mittag-Leffler function
$e_{\alpha}^v$	Generalized $\alpha$ -exponential function
$h$	Time delay
$I$	Identity Matrix
$k$	Variable denoting branch index of the Lambert W function
$S$	Solution matrix
$V$	Lyapunov Function
$W$	Symbolic representation for Lambert W function
$\sigma$	Sliding function
$\alpha$	Order of fractional order derivative
$\Theta$	Variable denoting an auxiliary matrix
$\Xi$	Variable denoting an $n \times n$ transformation matrix
$\forall$	For all
$\ x\ $	Euclidian norm of a vector $x$
$\lambda$	Eigenvalue of a matrix
$\omega$	Frequency