EXISTENCE AND MULTIPLICITY RESULTS FOR NONLOCAL ELLIPTIC CHOQUARD PROBLEMS

SUSHMITA RAWAT

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY DELHI
JANUARY 2024

EXISTENCE AND MULTIPLICITY RESULTS FOR NONLOCAL ELLIPTIC CHOQUARD PROBLEMS

by

SUSHMITA RAWAT

Department of Mathematics

Submitted
in fulfillment of the requirements of the degree of
Doctor of Philosophy

to the

Indian Institute of Technology Delhi
January 2024

Dedicated To My Family

Certificate

This is to certify that the thesis entitled **Existence and multiplicity results for nonlocal elliptic Choquard problems** submitted by **Sushmita Rawat** to the Indian Institute of Technology Delhi, for the award of the degree of **Doctor of Philosophy** is a record of the original bonafide research work carried out by her under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.
The results contained in this thesis have not been submitted in part or full to any other University or Institute for the award of any degree or diploma.

Date:
Place:

Prof. Konijeti Sreenadh
Institute Chair Professor
Department of Mathematics
Indian Institute of Technology Delhi
New Delhi 110016

Acknowledgements

This thesis represents the culmination of an arduous journey marked by persistent challenges, moments of exhaustion and relief. This voyage has been nothing short of life-altering, and I owe its successful completion to the unwavering support of many remarkable individuals. I extend my heartfelt gratitude to each person who, through their insightful comments, thoughtful questions, constructive criticisms, and both personal and academic encouragement, has left an indelible mark on this monumental work.

I wish to express my deepest and most profound appreciation to my remarkable supervisor, Professor K. Sreenadh, whose unwavering support and mentorship have been the cornerstone of my entire Ph.D. journey. From the very outset of my doctoral studies, Professor K. Sreenadh has been a beacon of guidance and inspiration, leading me through the academic rigors and nurturing my growth as a researcher. His commitment to my development extends far beyond the confines of academia. He has been an ardent advocate for my career aspirations, tirelessly seeking opportunities and providing invaluable insights to enhance my professional trajectory.

I want to express my deepest gratitude to Professor Jacques Giacomoni. His expertise and valuable insights have made a profound impact on the quality of my research. The numerous discussions and collaborative research initiatives we undertook were a significant source of inspiration and guidance.

I wish to acknowledge the contributions of the Student Research Council (SRC), Prof. Harish Kumar, Prof. Ananta Kumar Majee, and Prof. Amruta Mishra, for generously sharing their time and valuable insights. I am grateful to all the teachers who have contributed to my academic journey. I extend my gratitude to UGC(University Grants Commission) for the financial support that enabled me to conduct my research. I am thankful to Prof. Kapil Kumar Sharma (South Asian University) for generously offering their time to attend my JRF to SRF seminar. Additionally, I extend my gratitude to IIT Delhi for providing excellent facilities and financial support through RSTA to attend a conference in Italy.

I also extend my sincere appreciation to Dr. Divya Goel. She has been not just a collaborator but also a friend, offering assistance not only in academics but also in navigating the various challenges that cropped up during my Ph.D.

Special thanks are reserved for my colleague, Gurdev C. Anthal for their intellectual conversations and support throughout my Ph.D journey. I must also acknowledge the encouragement and thought-provoking discussions with my seniors, Dr. Deepak Kumar, whose guidance have been a constant source of clarity and solutions throughout my Ph.D, Dr. Sarika Goyal, and Dr. Reshmi Biswas.

My deepest gratitude also extends to my friends Disha, Amisha, Sonakshi, Hemila and Keshav for providing constant support throught my life. Their willingness to lend an ear and provide solutions to problems has been very helpful. I would also thank my dear friend Akash Vardhan for his unwavering support not only academically but also emotionally.

In addition to my academic associates, I extend my thanks to my friends Simran, Satyam, Shweta, Kavin, Himanshu, Ekta and all my batch mates, who shared the joyous moments during our time at IIT Delhi. I would also like to express my gratitude to my juniors Diksha, Nidhi and Shammi, for their valuable contributions and thoughtful discussions.

Lastly, I convey my deepest appreciation to my parents, Mr Harish Rawat and Mrs. Archana Rawat, and my brother Mayank for their unwavering support, boundless love, and constant push which made me achieve my goals and aspirations. I am forever indebted to them for giving me the wings to soar to great heights. I also extend my

heartfelt gratitude to my grandparents, Mr. Anand Rawat and Late Mrs. Saraswati Devi, for their invaluable prayers and consistent support throughout these years. I would also like to thank my family and cousins for all their love and support.
Above all, I humbly thank the Divine for their eternal blessings and the radiance they have cast upon my life. Thank you.

January 2024 *Sushmita Rawat*

Abstract

The central theme of this thesis is to examine the existence and multiplicity of weak solutions to elliptic problems for a class of Choquard equations driven by local or nonlocal operators. In recent decades, nonlocal partial differential equations have gained substantial attention due to their diverse applications across numerous fields, including engineering, biology, physics, and more. The thesis is divided into five chapters.

In Chapter 1, we offer a brief survey and establish the background for the study carried out in the following chapters. Moreover, we outline the structure of the thesis and emphasize its key contributions.

In Chapter 2, we investigate a class of critical Choquard nonlinearity characterized by a fractional Laplacian operator and featuring a perturbation term. Depending on the nature of the perturbation term, we analyze three Dirichlet boundary problems on a bounded domain and obtain the existence and multiplicity results for these critical problems. For the singular perturbation term, we establish the existence result for a class of Kirchhoff-Choquard problems. We obtain at least two positive weak solutions by employing appropriate minimization arguments and approximating the perturbed problem obtained through truncation of the singular term. We prove that each positive weak solution is bounded and satisfies Hölder's regularity. Next, we examine a class of Kirchhoff-Choquard problems with a regular subcritical perturbation term and obtain existence and multiplicity results using minimization arguments and the mountain-pass theorem. Lastly, we investigate a problem with a perturbation term consisting of sublinear and linear terms combined with critical Choquard nonlinearity, where the nonlinearity term satisfies the Ambrosetti-Prodi condition.

Chapter 3 explores a more general class of Choquard equations characterized by subcritical growth. Precisely, we investigate a non-autonomous problem governed by the p-Laplacian operator with a Kirchhoff-coefficient on the entire domain $\mathbb{R}^N$. Due to the Kirchhoff term and the subcritical growth of generalized Choquard nonlinearity, one can not directly apply variational methods. Moreover, the absence of the conditions like Ambrosetti-Rabinowitz and other assumptions, which help in establishing the geometric structure and boundedness of Palais-Smale sequences, lead us to work on the Pohožaev manifold. Mainly, we obtain the existence of positive high energy solutions for the non-autonomous problem. By utilizing the polarization argument, we establish the radial symmetry of solutions for the autonomous case, which is a topic of independent interest. Subsequently, employing the splitting lemma and the linking theorem, we attain the solutions for the non-autonomous case.

In Chapter 4, we explore the study of the existence of positive solitary wave solutions for generalized quasilinear Schrödinger equations in $\mathbb{R}^N$. We develop this theory for non-autonomous perturbations involving critical Choquard nonlinearity. Initially, we focus on the limiting case and employ the concentration-compactness principle. This leads to the attainment of a positive radial solution, which is also a ground state solution. By utilizing the delicate estimates concerning the critical Choquard term along with the critical level of the limiting functional, we establish the existence of a positive solution for the problem.

Chapter 5 of this thesis is dedicated to the exploration of Neumann problems involving critical Choquard nonlinearity. We examine the homogeneous Neumann boundary condition and attain a Brezis-Nirenberg type existence result. By assuming a suitable flatness condition on the boundary, we determine sharp conditions for the existence and non-existence of positive solutions to this class of problems. In order to demonstrate the existence of a solution, we need an appropriate version of Cherrier's inequality. We highlight that this extension is not straightforward and is of independent interest. Furthermore, we utilize this inequality and establish compactness of our Sobolev functional.

सार

इस थीसिस का केंद्रीय विषय स्थानीय या गैर-स्थानीय ऑपरेटरों द्वारा संचालित चोक्वार्ड समीकरणों के एक वर्ग के लिए अण्डाकार समस्याओं के कमजोर समाधानों के अस्तित्व और बहुलता की जांच करना है। हाल के दशकों में, गैर-स्थानीय आंशिक अंतर समीकरणों ने इंजीनियरिंग, जीव विज्ञान, भौतिकी और अन्य सहित कई क्षेत्रों में अपने विविध अनुप्रयोगों के कारण पर्याप्त ध्यान आकर्षित किया है। थीसिस पाँच अध्यायों में विभाजित है।

अध्याय 1 में, हम एक संक्षिप्त सर्वेक्षण प्रस्तुत करते हैं और निम्नलिखित अध्यायों में किए गए अध्ययन की पृष्ठभूमि स्थापित करते हैं। इसके अलावा, हम थीसिस की संरचना की रूपरेखा तैयार करते हैं और इसके प्रमुख योगदानों पर जोर देते हैं।

अध्याय 2 में, हम एक फ्रैक्शनल लाप्लासियन ऑपरेटर द्वारा विशेषता और एक गड़बड़ी शब्द की विशेषता वाले महत्वपूर्ण चोक्वार्ड नॉनलाइनरिटी के एक वर्ग की जांच करते हैं। गड़बड़ी शब्द की प्रकृति के आधार पर, हम एक बंधे हुए डोमेन पर तीन डिरिचलेट सीमा समस्याओं का विश्लेषण करते हैं और इन महत्वपूर्ण समस्याओं के लिए अस्तित्व और बहुलता परिणाम प्राप्त करते हैं। एकवचन गड़बड़ी शब्द के लिए, हम किरचॉफ-चोक्वार्ड समस्याओं के एक वर्ग के लिए अस्तित्व परिणाम स्थापित करते हैं। हम उचित न्यूनतमकरण तर्कों को नियोजित करके और एकवचन शब्द के काट-छांट के माध्यम से प्राप्त परेशान समस्या का अनुमान लगाकर कम से कम दो सकारात्मक कमजोर समाधान प्राप्त करते हैं। हम साबित करते हैं कि प्रत्येक सकारात्मक कमजोर समाधान सीमित है और होल्डर की नियमितता को संतुष्ट करता है।

इसके बाद, हम एक नियमित सबक्रिटिकल गड़बड़ी शब्द के साथ किरचॉफ-चोक्वार्ड समस्याओं के एक वर्ग की जांच करते हैं और न्यूनतमकरण तर्कों और माउंटेन-पास प्रमेय का उपयोग करके अस्तित्व और बहुलता परिणाम प्राप्त करते हैं। अंत में, हम एक गड़बड़ी शब्द के साथ एक समस्या की जांच करते हैं जिसमें उप-रैखिक और रैखिक शब्दों को महत्वपूर्ण चोक्वार्ड नॉनलाइनरिटी के साथ जोड़ा जाता है, जहां नॉनलाइनरिटी शब्द एम्ब्रोसेटी-प्रोडी स्थिति को संतुष्ट करता है।

अध्याय 3 चॉक्वार्ड समीकरणों के एक अधिक सामान्य वर्ग की खोज करता है जो उप-महत्वपूर्ण विकास की विशेषता है। सटीक रूप से, हम संपूर्ण डोमेन आर^एन पर किरचॉफ-गुणांक के साथ पी-लाप्लासियन ऑपरेटर द्वारा नियंत्रित एक गैर-स्वायत्त समस्या की जांच करते हैं। किरचॉफ शब्द और सामान्यीकृत चोक्वार्ड गैर-रैखिकता की उप-महत्वपूर्ण वृद्धि के कारण, कोई सीधे तौर पर परिवर्तनशील तरीकों को लागू नहीं कर सकता है। इसके अलावा, एम्ब्रोसेटी-राबिनोविट्ज़ और अन्य मान्यताओं जैसी स्थितियों की अनुपस्थिति, जो कि पैलैस-स्माले अनुक्रमों की ज्यामितीय संरचना और सीमा को स्थापित करने में मदद करती है, हमें पोहोज़ेव मैनिफोल्ड पर काम करने के लिए प्रेरित करती है। मुख्य रूप से, हम गैर-स्वायत्त समस्या के लिए सकारात्मक उच्च-ऊर्जा समाधान के अस्तित्व को प्राप्त करते हैं। ध्रुवीकरण तर्क का उपयोग करके, हम स्वायत्त मामले के लिए समाधान की रेडियल समरूपता स्थापित करते हैं, जो स्वतंत्र रुचि का विषय है। इसके बाद, विभाजन प्रमेयिका और लिंकिंग प्रमेय का उपयोग करते हुए, हम गैर-स्वायत्त मामले के समाधान प्राप्त करते हैं।

अध्याय 4 में, हम R^N में सामान्यीकृत क्वासिलिनियर श्रोडिंगर समीकरणों के लिए सकारात्मक एकान्त तरंग समाधानों के अस्तित्व के अध्ययन का पता लगाते हैं। हम इस सिद्धांत को महत्वपूर्ण चोक्वार्ड गैर-रैखिकता से जुड़े गैर-स्वायत्त गड़बड़ी के लिए विकसित करते हैं। प्रारंभ में, हम सीमित मामले पर ध्यान केंद्रित करते हैं और एकाग्रता-कॉम्पैक्टनेस सिद्धांत को नियोजित करते हैं।

इससे एक सकारात्मक रेडियल समाधान की प्राप्ति होती है, जो एक ग्राउंड-स्टेट समाधान भी है। सीमित कार्यात्मकता के महत्वपूर्ण स्तर के साथ-साथ महत्वपूर्ण चोक्वार्ड शब्द से संबंधित नाजुक अनुमानों का उपयोग करके, हम समस्या के लिए एक सकारात्मक समाधान के अस्तित्व को स्थापित करते हैं।

इस थीसिस का अध्याय 5 महत्वपूर्ण चोक्कार्ड गैर-रैखिकता से जुड़ी न्यूमैन समस्याओं की खोज के लिए समर्पित है। हम सजातीय न्यूमैन सीमा स्थिति की जांच करते हैं और ब्रेज़िस-निरेनबर्ग प्रकार के अस्तित्व परिणाम प्राप्त करते हैं। सीमा पर उपयुक्त समतलता की स्थिति मानकर, हम समस्याओं के इस वर्ग के सकारात्मक समाधानों के अस्तित्व और गैर-अस्तित्व के लिए तीव्र स्थितियाँ निर्धारित करते हैं। किसी समाधान के अस्तित्व को प्रदर्शित करने के लिए, हमें चेरियर की असमानता के एक उपयुक्त संस्करण की आवश्यकता है। हम इस बात पर प्रकाश डालते हैं कि यह विस्तार सीधा नहीं है और स्वतंत्र हित का है। इसके अलावा, हम इस असमानता का उपयोग करते हैं और हमारे सोबोलेव कार्यात्मकता की कॉम्पैक्टनेस स्थापित करते हैं।

Table of Contents

Certificate i

Acknowledgements iii

Abstract v

List of Symbols ix

1 Introduction 1

1.1 Nonlocal elliptic problems . . . 4
1.2 Critical growth problems . . . 8
1.3 Kirchhoff: Nonlocal coefficient . . . 10
1.4 Singular problems . . . 11
1.5 Organization of the Thesis . . . 12
1.5.1 Critical Choquard problems with subcritical perturbation . . . 13
1.5.2 High energy solutions for a class of quasilinear Choquard problem . . . 17
1.5.3 Quasilinear Schrödinger equation . . . 21
1.5.4 Critical Neumann problem . . . 23

2 Critical Choquard problems with subcritical perturbation 25

2.1 Introduction . . . 25
2.2 Singular perturbation . . . 26
2.2.1 Preliminaries . . . 26
2.2.2 Regularity of solutions of (S_λ) . . . 28
2.2.3 Existence of first solution . . . 31
2.2.4 Perturbed problem $(S_{n,\lambda})$ . . . 36
2.2.5 Existence of Second Solution . . . 43
2.3 Regular perturbation . . . 45
2.3.1 Preliminaries . . . 47
2.3.2 Technical lemmas . . . 47
2.3.3 Proof of Theorem 2.3.1 and 2.3.2 . . . 57
2.3.4 Multiplicity result for $\theta \geq 2^*_{\mu,s}$ and $2 < q < 2^*_s$ . . . 60
2.3.5 Multiplicity result for sublinear case $1 < q \leq 2$ . . . 63
2.4 Asymmetric Problem . . . 71
2.4.1 Preliminaries . . . 72
2.4.2 Palais-Smale condition . . . 73
2.4.3 Constant sign solutions . . . 76
2.4.4 Linking geometry . . . 80
2.4.5 Third solution . . . 87
2.5 Conclusions . . . 93

3 High energy solutions for a class of quasilinear Choquard problem 95
3.1 Introduction . . . 95
3.2 Preliminaries and Main results . . . 96
3.3 Technical lemmas . . . 99
3.4 Radial ground states of limiting problem . . . 103
3.5 Compactness of high energy (PS)-sequences . . . 107
3.6 Proof of Theorem 3.2.2 . . . 117
3.7 Conclusions . . . 120

4 Quasilinear Schrödinger equation 121
4.1 Introduction . . . 121
4.2 Preliminaries . . . 124
4.3 Technical lemmas . . . 124
4.4 Limiting equation at infinity . . . 128
4.5 Proof of Theorem 4.1.1 . . . 138
4.6 Conclusions . . . 145

5 Critical Neumann problem 147
5.1 Introduction . . . 147
5.2 Preliminaries . . . 148
5.3 Subcritical case $2 \leq p < 2^*_\mu$ . . . 150
5.4 Critical case $p = 2^*_\mu$ . . . 151
5.5 Conclusions . . . 159

References 161

List of Publications 171

Curriculum Vitae 173

List of Symbols

Symbol	**Meaning**
$\mathbb{R}$	Set of real numbers.
$\mathbb{N}$	Set of natural numbers.
$\|A\|$	Lebesgue measure of a set $A \subset \mathbb{R}^N$.
$B_\epsilon(x)$	Ball of radius ϵ and centered at $x \in \mathbb{R}^N$.
$\partial\Omega$	Boundary of Ω
$C^\infty(\Omega)$	Set of infinitely differentiable functions in Ω.
$C_0^\infty(\Omega)$	Set of infinitely differentiable functions with compact support in Ω.
$\|f\|_{L^p}$	Norm of f in L^p.
u^+	$\max\{u, 0\}$.
u^-	$\max\{-u, 0\}$
$\text{supp}(u)$	Closure of the set $\{x : u(x) \neq 0\}$ in $\mathbb{R}^N$.
$\mathbb{R}^N_+$	$\{x \in \mathbb{R}^N : x_N > 0\}$.
$*$	the standard convolution operator.
Δ	Laplacian.
Δ_p	p-Laplacian.
$H^{-1}(\Omega)$	Dual space of $H_0^1(\Omega)$.
$\|\cdot\|_*$	Norm in dual space.
$H_0^1(\Omega)$	Closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.