STOCHASTIC DEGENERATE/FRACTIONAL CONSERVATION LAWS: WELL-POSEDNESS, NUMERIC AND LARGE DEVIATION PRINCIPLE

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Stochastic degenerate/fractional conservation laws: well-posedness, numeric and large deviation principle

by

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Dedicated to
my parents

Mr Mohini Mohan Behera and Mrs Sailendri Behera

and my niece

Parnika

Certificate

This is to certify that the thesis entitled **Stochastic degenerate/fractional conservation laws: well-posedness, numeric and large deviation principle** submitted by **Mr. Soumya Ranjan Behera** to **Indian Institute of Technology Delhi**, for the award of the degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by him under my supervision and guidance. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree. The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

New Delhi
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New Delhi Soumya Ranjan Behera

Abstract

The goal of this thesis is to explore the analytical and numerical aspects of Cauchy and Dirichlet problems related to degenerate parabolic-hyperbolic partial differential equations and fractional conservation laws driven by Lévy noise. The first Chapter delved into a brief survey of the literature on deterministic conservation laws on both bounded and unbounded domains. In addition, we also review the available results on stochastic conservation laws. At the end of this chapter, we give an overview of our contribution to the thesis.

In Chapter 2, we study the convergence of approximate solutions generated by the operator splitting scheme towards the unique BV entropy solution for fractional conservation laws driven by Lévy noise. A variant of classical Kružkov's doubling of variables approach plays an important role in our convergence analysis. Finally, the convergence analysis is illustrated by several numerical examples. Chapter 3 is devoted to study the large and moderate deviation principles (LDP and MDP) and the central limit theorem (CLT) for stochastic fractional conservation laws in the kinetic formulation framework. The weak convergence method and doubling the variables technique play crucial roles in this analysis. Moreover, the LDP is established in a more general setting (square-integrable periodic initial datum). At the same time, since the kinetic solution of the underlying problem predominantly sits in an irregular space, we prove CLT and MDP for a constant initial data. We delved into the existence and uniqueness of the renormalized entropy solution for degenerate parabolic-hyperbolic Cauchy problem perturbed by a multiplicative Lévy noise with general L^1-data in Chapter 4. By using a suitable approximation procedure based on the vanishing viscosity technique and bounded data, we prove the existence of a renormalized entropy solution to the underlying problem. The uniqueness of the solution is settled by adapting Kružkov's doubling the variables technique in the presence of noise.

In Chapters 5 and 6, we analyzed the well-posedness of entropy solution for homogeneous and non-homogeneous Dirichlet problems for degenerate parabolic-hyperbolic PDE perturbed by Lévy noise. We consider the homogeneous Dirichlet problem in Chapter 5. Here, the main arguments were based on Kružov's semi-entropy formulation and obtaining local and global Kato's inequalities, allowing comparison principle in L^1 and thereby existence and uniqueness of entropy solution. In Chapter 6, we work on establishing the well-posedness analysis for the non-homogeneous Dirichlet problem. To get the uniqueness of entropy solutions, we are using a version of the doubling variables method within a stochastic framework. Our proof entails a nuanced examination of boundary values of solutions, requiring the development of innovative techniques to leverage the existence of normal weak traces for divergence measure fields in the noise environment. Moreover, a generic method is developed to prove the well-posedness theory for the associated viscous problems with the help of Aldous tightness criteria and Jakubowski's version of the Skorokhod theorem through an approximation process consisting of a bi-Laplacian term.

सारांश

इस शोध का उद्देश्य लेवी शोर से प्रेरित विलोमित पराबोलिक-हाइपरबोलिक आंशिक अंतर समीकरणों और भिन्नात्मक संरक्षण नियमों से संबंधित कॉशी और डिरिखले समस्याओं के विश्लेषणात्मक और संख्यात्मक पहलुओं का अन्वेषण करना है। पहले अध्याय में सीमाबद्ध और असीमित क्षेत्रों पर निर्धारक संरक्षण नियमों पर साहित्य का संक्षिप्त सर्वेक्षण किया गया है। इसके अतिरिक्त, हम स्टोकेस्टिक संरक्षण नियमों पर उपलब्ध परिणामों की समीक्षा करते हैं। इस अध्याय के अंत में, हम शोध में हमारे योगदान का अवलोकन प्रस्तुत करते हैं।

दूसरे अध्याय में, हम लेवी शोर से प्रेरित भिन्नात्मक संरक्षण नियमों के लिए ऑपरेटर स्प्लिटिंग योजना द्वारा उत्पन्न निकटतम हलों के अद्वितीय BV एंट्रॉपी हल की ओर अभिसरण का अध्ययन करते हैं। हमारे अभिसरण विश्लेषण में क्रुज़कोव के क्लासिकल डबलिंग ऑफ वेरिएबल्स दृष्टिकोण का एक संशोधित रूप महत्वपूर्ण भूमिका निभाता है। अंत में, अभिसरण विश्लेषण को कई संख्यात्मक उदाहरणों द्वारा प्रदर्शित किया गया है। तीसरा अध्याय स्टोकेस्टिक भिन्नात्मक संरक्षण नियमों के लिए बड़े और मध्यम विचलन सिद्धांत (LDP और MDP) और केंद्रीय सीमा प्रमेय (CLT) के अध्ययन के लिए समर्पित है, जो काइनेटिक संरचना ढांचे में किया गया है। इस विश्लेषण में कमजोर अभिसरण विधि और डबलिंग ऑफ वेरिएबल्स तकनीक महत्वपूर्ण भूमिका निभाते हैं। इसके अलावा, LDP को अधिक सामान्य परिप्रेक्ष्य में स्थापित किया गया है, जिसमें वर्ग-एकीकृत आवधिक प्रारंभिक डेटा शामिल है। साथ ही, चूंकि अंतर्निहित समस्या का काइनेटिक समाधान मुख्यतः एक अनियमित स्थान में स्थित होता है, हम स्थिर प्रारंभिक डेटा के लिए CLT और MDP सिद्ध करते हैं। चौथे अध्याय में, हम गुणकात्मक लेवी शोर से प्रभावित सामान्य L^1-डेटा के साथ विलोमित पराबोलिक-हाइपरबोलिक कॉशी समस्या के लिए पुनर्नियोजित एंट्रॉपी समाधान के अस्तित्व और अद्वितीयता का अध्ययन करते हैं। वैनिशिंग विस्कोसिटी तकनीक और सीमाबद्ध डेटा पर आधारित एक उपयुक्त दृष्टिकोण प्रक्रिया का उपयोग करते हुए, हम अंतर्निहित समस्या के लिए पुनर्नियोजित एंट्रॉपी समाधान का अस्तित्व सिद्ध करते हैं। समाधान की अद्वितीयता को शोर की उपस्थिति में क्रुज़कोव की डबलिंग ऑफ वेरिएबल्स तकनीक को अपनाकर स्थापित किया गया है।

पाँचवें और छठे अध्याय में, हमने लेवी शोर द्वारा प्रभावित विलोमित पराबोलिक-हाइपरबोलिक आंशिक अंतर समीकरणों के लिए समरूप और असमरूप डिरिखले समस्याओं के एंट्रॉपी समाधान की बेहतर सुसंगतता का विश्लेषण किया। पाँचवें अध्याय में, हम समरूप डिरिखले समस्या का अध्ययन करते हैं। यहाँ, मुख्य तर्क क्रुज़कोव के अर्ध-एंट्रॉपी सूत्रीकरण और स्थानीय तथा वैश्विक काटो असमानताओं को प्राप्त करने पर आधारित थे, जो L^1 में तुलना सिद्धांत की अनुमति देते हैं और इस प्रकार एंट्रॉपी समाधान के अस्तित्व और अद्वितीयता को सिद्ध करते हैं। छठे अध्याय में, हमने असमरूप डिरिखले समस्या के लिए सुसंगतता सिद्धांत स्थापित करने पर काम किया। एंट्रॉपी समाधानों की अद्वितीयता प्राप्त करने के लिए, हम स्टोकेस्टिक ढांचे में डबलिंग ऑफ वेरिएबल्स पद्धति के एक संस्करण का उपयोग कर रहे हैं। हमारे प्रमाण में समाधान के सीमा मानों की सूक्ष्म परीक्षा शामिल है, जिसके लिए शोर के वातावरण में विचलन माप क्षेत्रों के लिए सामान्य कमजोर ट्रेस के अस्तित्व का लाभ उठाने हेतु नई तकनीकों का विकास करना आवश्यक था। इसके अलावा, संबंधित विस्कस समस्याओं के लिए सुसंगतता सिद्धांत को सिद्ध

करने हेतु एक सामान्य विधि विकसित की गई है, जिसमें एल्डस की सघनता मानदंड और याकुबोव्स्की के स्कोरोखोद प्रमेय के संस्करण का उपयोग किया गया है, जिसे द्वि-लैप्लेशियन शब्द के साथ एक अभिसारणी प्रक्रिया के रूप में लागू किया गया है।

Contents

List of Figures