A STUDY OF WEYL TRANSFORM AND ITS MULTIPLIER THEORY: SCALAR AND VECTOR ANALOGUE

RITIKA SINGHAL

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY DELHI

JULY2024

A study of Weyl transform and its multiplier theory: Scalar and vector analogue

by

RITIKA SINGHAL
DEPARTMENT OF MATHEMATICS

Submitted

in fulfilment of the requirements of the degree of Doctor of Philosophy

to the

Indian Institute of Technology Delhi

JULY 2024

Dedicated to my teachers & family

Certificate

This is to certify that the thesis entitled **A study of Weyl transform and its multiplier theory: Scalar and vector analogue** submitted by **Ms. Ritika Singhal** to the **Indian Institute of Technology Delhi**, for the award of the Degree of **Doctor of Philosophy**, is a record of the original bonafide research work carried out by her under my guidance and supervision. The thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree or diploma.

July 2024
New Delhi

Dr. N. Shravan Kumar
Associate Professor
Dept. of Mathematics
IIT Delhi, New Delhi

Acknowledgements

ॐ गुरुर्ब्रह्मा गुरुर्विष्णुः गुरुर्देवो महेश्वरः । गुरुरेव परं ब्रह्म तस्मै श्रीगुरवे नमः ॥

A teacher is the Creator (Brahma), the Preserver(Vishnu), and the Destroyer (Maheshwara). The teacher is the embodiment of the supreme lord, salutations to that revered teacher.

First and foremost, I sincerely thank Dr. N Shravan Kumar, my supervisor, for his invaluable feedback and guidance during my PhD journey. There's no way I would have embarked on this journey without him being the light bearer. His knowledge, encouragement, and perseverance have greatly influenced this work and improved my scholarly experience. He taught me the importance of working hard and the need to be patient in difficult circumstances.

I extend my appreciation to all my teachers who have contributed to my education and personal growth. I want to express my gratitude to my Student Research Committee (SRC) members, Prof. S. Sampath, Dr. A. Dasgupta, Prof. Ravishankar, and Prof. S.D. Joshi, for their insightful feedback throughout my PhD. Special thanks to Prof. A. Mehra, Head of the Mathematics Department, for inspiring me to achieve greater heights and for her support. I want to acknowledge my institute, IIT Delhi, for providing me with this platform. The resources and facilities provided have been indispensable in completing this thesis. The financial support offered by IIT Delhi through the GATE fellowship and RSTA to present my work at international conferences is greatly acknowledged.

I will always be grateful to my siblings Piyush, Deepak, and Priyanka for inspiring me to pursue a PhD, being there for me as friends, and adding cheerfulness to my life. I am grateful for my parents' unwavering support, patience, and understanding. I am forever indebted for their sacrifices, encouragement, and belief in my abilities. To my mother, your presence in my life has been a source of strength and inspiration.

From the start of my PhD journey, I've been embraced by the unwavering support of friends like Abhilash Tushir, ever just a call away; Anshul Prajapati, a constant source of calm amidst my anxieties; and Sachin Jakhar, who unfailingly uplifted my spirits. To my friends, Bhawna, Shweta, Himanshu and Satyam, staying in IIT would have been lifeless without you. I am grateful to my colleagues Praveen, Gurdev, Sushmita, Astha, Varagpriya and Gyan for their support. I am indebted to my academic siblings, Kanupriya, Arvish, Prerna and Sridharan, whose presence made the most challenging phase of my PhD journey easier. Thanks to my labmates, Navneet, Kavin, Lalit, Naveen, Anjali and Anuj, for creating a homely environment around me. Special mention goes to Dr. Sushil Singla and Dr. Bharat Talwar, whose guidance has been invaluable to me.

Lastly, through this platform, I would like to thank my closest friends, Kriti, Shweta, Jyoti, Shrey, Anubhav and Jaya, who have always been my cheerleaders. All others who have silently supported my journey, I am thankful to them.

This work would not have been possible without the support and encouragement of all those mentioned above. Thank you for being an integral part of this journey. I extend my heartfelt gratitude to the Almighty and the vast universe for bestowing me the opportunity, strength, and empowerment to complete this journey.

New Delhi

July, 2024 Ritika Singhal

Abstract

This thesis is devoted to the study of certain aspects of the Weyl transform associated to locally compact abelian groups. It is a detailed study of the Weyl transform of the vector-valued functions and the multiplier theory related to it.

We start by introducing the notion of the Weyl transform for the Bochner integrable function and study the boundedness of the operator. We discussed the classical Hausdorff-Young inequality in this context. We also studied the Weyl transform of vector measures and functions which are integrable with respect to a vector measure, and showed the failure of the Reimann-Lebesque lemma. We also provided a sufficient condition for the Reimann-Lebesque lemma to hold true.

Later, we prove several versions of the classical Paley inequality for the Weyl transform. As for some applications, we prove a version of the Hörmander's multiplier theorem to discuss L^p-L^q boundedness of the Weyl multipliers and prove the Hardy-Littlewood inequality. We also consider the vector-valued version of the inequalities of Paley, Hausdorff-Young, and Hardy-Littlewood and their relations. Eventually, we prove Pitt's inequality for the Weyl transform.

Ultimately, we will characterize vector-valued Weyl multipliers for the pair L^1-L^p, $1 \leq p < \infty$, assuming that the underlying Banach algebra has a bounded approximate identity.

सार

इस शोध प्रबंध में स्थानीय सहंत एबेलियन समूहों से संबंधित वायल ट्रांसफॉर्म के कुछ पहलुओं के अध्ययन पर ध्यान केंद्रित किया गया है। यह सदिश–मूल्य फलनों के वायल ट्रांसफॉर्म और इससे संबंधित गुणक सिद्धांत का विस्तृत अध्ययन है। हम बोकनर समाकलनीय फलन के लिए वायल ट्रांसफॉर्म की अवधारणा का परिचय देकर प्रारंभ करते हैं और संचालक की सीमाबद्धता का अध्ययन करते हैं। इस संदर्भ में हमने पारंपरिक हाउज़डॉर्फ़–यंग असमानता पर चर्चा की। हमने सदिश माप और उन फलनों का वायल ट्रांसफॉर्म भी अध्ययन किया जो सदिश माप के संबंध में समाकलनीय हैं और रीमैन–लेबेग उपवाक्य की असफलता को दर्शाया। हमने रीमैन–लेबेग उपवाक्य के सत्य होने के लिए एक पर्याप्त शर्त भी प्रदान की।

बाद में, हम वायल ट्रांसफॉर्म के लिए पारंपरिक पैली असमानता के कई संस्करणों को सिद्ध करते हैं। कुछ अनुप्रयोगों के लिए, हम वायल गुणकों की L^p–L^q सीमाबद्धता पर चर्चा करने के लिए होर्मांडर के गुणक प्रमेय का एक संस्करण सिद्ध करते हैं और हार्डी–लिटिलवुड असमानता को सिद्ध करते हैं। हमने पैली, हाउज़डॉर्फ़–यंग, और हार्डी–लिटिलवुड की असमानताओं के सदिश–मूल्य संस्करण और उनके संबंधों पर भी विचार किया। अंततः, हमने वायल ट्रांसफॉर्म के लिए पिट की असमानता को सिद्ध किया।

अंत में, हम सदिश–मूल्य वायल गुणकों को युग्म L^1–$L^p, 1 \leq p < \infty$ के लिए अभिलाक्षित करेंगे, यह मानते हुए कि अंतर्निहित बानाच बीजगणित के पास एक सीमाबद्ध अनुकरणीय पहचान है।

Contents

List of Symbols

Symbol	**Meaning**
$\forall$	for all
$=$	equal to
$\neq$	not equal to
$\in$	belongs to
$\notin$	does not belong
$\subset$	subset or equal
$\cup$, $\cap$	union, intersection
$\emptyset$	empty set
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers

G	the abelian group
m_G	the Haar measure on G
$\widehat{G}$	the dual group of G
$S(G)$	the space of complex-valued simple functions on G
$C(G)$	the space of complex-valued continuous functions on G
$M(G)$	the space of complex Radon measures on G
$\widehat{f}$	the Fourier transform of f
w.r.t.	with respect to
a.e.	almost everywhere